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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On generalized Weber and Clebsch transformations

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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SUITABLE generalizations of the Weber and Clebsch transformations of the hydrodynamic equations are introduced which have some bearing in the treatment of the inverse problem of Lagrangian field theory. In particular these generalizations open the way to equivalence proofs for several Lagrangians proposed in the realm of ideal (magneto-)hydrodynamics. This means that the Euler–Lagrange equations corresponding to these Lagrangians do not only imply but are also implied by the original field equations of the systems under study.

1. Introduction

GIVEN A SET of field equations for a dynamical system, the inverse problem of Lagrangian field theory deals with the question whether one can find an action principle for it. In other words, it is examined whether Lagrangians can be constructed whose Euler - Lagrange equations are equivalent to a given set of field equations.

Concerning the "Eulerian" field equations of ideal fluid flow in hydrodynamics, magnetohydrodynamics, and plasma dynamics, a large number of proposed Lagrangians can be found in the literature. To quote only a few surveys of the whole field, see [1–6]. However, in several cases it has only been shown that the respective Euler - Lagrange equations imply the validity of the original hydrodynamical flow equations. But to establish full equivalence between the Euler - Lagrange equations and the original field equations, it is also necessary to examine whether the Euler - Lagrange equations are capable of describing *all* possible solutions of the original equations.

Up to now, complete equivalence proofs for certain Lagrangians have been given for the cases of the barotropic and the non-barotropic ideal fluid [1, 7]. The main tools involved in these proofs are representations of vector fields in terms of special potential classes as well as the so-called Weber and Clebsch transformations of the hydrodynamic equations.

One aim of this paper is to show that – employing suitable generalizations of these Weber and Clebsch transformations – complete equivalence proofs can also be given for many other Lagrangians in hydrodynamics, magnetohydrodynamics, and plasma dynamics. The class of systems which can be treated on more or

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less the same footing is rather large and comprises e. g. charged ideal fluids in external electromagnetic fields and ideal magnetohydrodynamic fluids with infinite conductivity. Even a certain case of fluid flow in porous media – obeying a nonstationary extension of Darcy's law – turns out to be covered. The present paper is organized as follows:

Section 2 is devoted to fixing of the notation. In Sec. 3, the original Weber and Clebsch transformations for barotropic fluids are shortly revisited and their connection to the solution of the inverse problem of Lagrangian field theory is pointed out. In Sec. 4 we then introduce suitable generalizations of the Weber and the Clebsch transformations, starting from a generalized form of the Euler equation. In the next section it is demonstrated that there is a considerable number of hydrodynamical systems whose respective Euler equations can be cast into such a generalized form. The paper is then concluded with a list of Lagrangians for these systems for which equivalence proofs are now available.

Due to limitations of space, not all the – sometimes lengthy – derivations can be given here. Only the generalized Weber and Clebsch transformations are treated in some detail, whereas we have to restrict ourselves to summary remarks in the remaining sections. The full length considerations can be found elsewhere [6].

2. Notation

The trajectories of the material points of the continuum are given as follows:

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)\,. \tag{2.1}$$

Here t means time and $\mathbf{x}_0$ denotes the material (Lagrangian) coordinates:

$$\mathbf{x}_0 = \mathbf{x}(\mathbf{x}_0, 0)\,. \tag{2.2}$$

We require (2.1) to be invertible, thus leading to the "index field"

$$\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x}, t). \tag{2.3}$$

The "Eulerian" velocity field $\mathbf{u}(\mathbf{x}, t)$ is given as

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{x}_0, t)\Big|_{\mathbf{x}_0=\mathbf{x}_0(\mathbf{x},t)}. \tag{2.4}$$

Generally, the fields of the form $\varepsilon(\mathbf{x}, t)$ – i.e., described as functions of $\mathbf{x}$ and t – are called "Eulerian fields". They give rise to the corresponding "Lagrangian fields" $\varepsilon^{(L)}$ depending on $\mathbf{x}_0$ and t:

$$\varepsilon^{(L)}(\mathbf{x}_0, t) = \varepsilon(\mathbf{x}(\mathbf{x}_0, t), t)\,. \tag{2.5}$$

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The following abbreviation for the substantial time derivative is used in this paper:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\,. \tag{2.6}$$

It mainly comes into the play by means of the "local transport theorem"

$$\frac{\partial}{\partial t}\varepsilon^{(L)}(\mathbf{x}_0,t)\bigg|_{\mathbf{x}_0=\mathbf{x}_0(\mathbf{x},t)} = \frac{D}{Dt}\varepsilon(\mathbf{x},t)\,. \tag{2.7}$$

3. Weber and Clebsch transformations revisited

In this section we give a rather condensed review of the Weber and Clebsch transformations for the barotropic ideal fluid, i.e., for a fluid where the pressure p is a function of the mass density ϱ only. For details the reader is referred to §§ 15, 167 of [8] and § 29 of [1].

Our starting point is the Euler equation

$$\frac{D}{Dt}\mathbf{u} = -\nabla\left(\int\frac{dp(\varrho)}{\varrho} + U\right) = -\nabla(P+U). \tag{3.1}$$

After transition to the Lagrangian picture (Lagrangian equations of motion), one can derive the so-called "Weber transformation of the hydrodynamic equations"

$$\sum_k u_k^{(L)}(\mathbf{x}_0,t)\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0,t) = u_i^{(L)}(\mathbf{x}_0,0) + \frac{\partial}{\partial x_{0i}}\phi^{(L)}(\mathbf{x}_0,t) \tag{3.2}$$

with

$$\phi^{(L)}(\mathbf{x}_0,t) = \int\limits_0^t \left(\frac{1}{2}\mathbf{u}^{(L)}(\mathbf{x}_0,t')^2 - P^{(L)}(\mathbf{x}_0,t') - U^{(L)}(\mathbf{x}_0,t')\right)dt'\,. \tag{3.3}$$

Transition back to the Eulerian picture implies the "Lin representation" of the velocity field

$$\mathbf{u}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t) + \sum_i \alpha_i(\mathbf{x},t)\nabla x_{0i}(\mathbf{x},t) \tag{3.4}$$

with

$$\boldsymbol{\alpha}(\mathbf{x},t) = \mathbf{u}^{(L)}(\mathbf{x}_0(\mathbf{x},t),0)\,, \qquad \phi(\mathbf{x},t) = \phi^{(L)}(\mathbf{x}_0(\mathbf{x},t),t)\,. \tag{3.5}$$

Due to (2.7), $\boldsymbol{\alpha}$, $\mathbf{x}_0$, ϕ solve the following equations:

$$\frac{D}{Dt}\boldsymbol{\alpha} = 0\,, \qquad \frac{D}{Dt}\mathbf{x}_0 = 0\,, \qquad \frac{D}{Dt}\phi = \frac{1}{2}\mathbf{u}^2 - P - U\,. \tag{3.6}$$

Note that the Euler equation (3.1) can be rederived from (3.4), (3.6).

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Now, what has this to do with the inverse problem of Lagrangian field theory? The Euler-Lagrange equations of the so-called "Lin Lagrangian" [9]

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho P(\varrho) + p(\varrho) - \varrho U + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\boldsymbol{\alpha}\cdot\frac{D}{Dt}\mathbf{x}_0 \tag{3.7}$$

can be shown to include (3.4), (3.6), whereas the Euler equation is missing. The above considerations now show that (3.4), (3.6) are an equivalent substitute for the original Euler equation.

The Lin representation of the velocity field is globally valid. If one only looks for a local representation, one can achieve a much simpler expression for **u**. This is due to the fact that – at least locally – any vector field **A** can be represented in terms of the "Clebsch potentials":

$$\mathbf{A} = \nabla\psi_0 + \alpha_0\nabla\beta_0\,. \tag{3.8}$$

For a proof see e.g. [10] and the references quoted there.

In particular, $\mathbf{u}^{(L)}(\mathbf{x}_0, 0)$ can thus be represented as

$$u_i^{(L)}(\mathbf{x}_0, 0) = \frac{\partial}{\partial x_{0i}}\psi_0(\mathbf{x}_0) + \alpha_0(\mathbf{x}_0)\frac{\partial}{\partial x_{0i}}\beta_0(\mathbf{x}_0). \tag{3.9}$$

Insertion of this representation into the Weber transformation and subsequent transition to the Eulerian picture implies

$$\mathbf{u} = \nabla\tilde{\phi} + \alpha\nabla\beta \tag{3.10}$$

with

$$\frac{D}{Dt}\tilde{\phi} - \frac{\mathbf{u}^2}{2} + P + U = 0\,, \qquad \frac{D}{Dt}\alpha = 0\,, \qquad \frac{D}{Dt}\beta = 0\,. \tag{3.11}$$

Again, the Euler equation can be rederived from these expressions.

Equations (3.10), (3.11) are included in the set of Euler-Lagrange equations of the "Davydov Lagrangian" [11]

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho P(\varrho) + p(\varrho) - \varrho U + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\alpha\frac{D}{Dt}\beta \tag{3.12}$$

which arises from (3.7) by replacing $\boldsymbol{\alpha}\cdot(D/Dt)\mathbf{x}_0$ with the simplified expression $\alpha(D/Dt)\beta$. Therefore, the Euler equation is locally equivalently substituted within the Euler-Lagrange equations of (3.12).

REMARK. In the literature instead of the Davydov Lagrangian one often finds the "Bateman Lagrangian" [12]

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho P(\varrho) + p(\varrho) - \varrho U - \varrho\frac{D}{Dt}\phi - \varrho\alpha\frac{D}{Dt}\beta \tag{3.13}$$

which differs from the Davydov Lagrangian only by a 4-divergence and thus gives rise to the same Euler-Lagrange equations.

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4. Generalized Weber and Clebsch transformations

Our starting points in this section are field quantities $\mathbf{Y}(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ which are supposed to satisfy the "generalized Euler equation"

$$\frac{D}{Dt}\mathbf{Y} + \sum_j Y_j \nabla u_j = \nabla\psi\,. \tag{4.1}$$

Just as the usual Euler equation (3.1) for a barotropic fluid in an external potential field serves as a starting point for the derivation of the original Weber transformation, the generalized Euler equation (4.1) will be shown to imply a certain "generalized Weber transformation". The following steps rather closely resemble those for the barotropic fluid which can be recovered as a special case by putting $\mathbf{Y} = \mathbf{u}$ and $\psi = \mathbf{u}^2/2 - P - U$.

Reexpressing (4.1) in Lagrangian coordinates, with (2.7) we get

$$\frac{\partial}{\partial t}\mathbf{Y}^{(L)}(\mathbf{x}_0, t) + \sum_j Y_j^{(L)}(\mathbf{x}_0, t)\nabla u_j(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}(\mathbf{x}_0,t)} = \nabla\psi(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}(\mathbf{x}_0,t)}. \tag{4.2}$$

This implies the "generalized Lagrangian equations of motion"

$$\begin{aligned}\sum_k \frac{\partial}{\partial t}Y_k^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t) & \\ + \sum_{j,k} Y_j^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_k}u_j(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}(\mathbf{x}_0,t)} &\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t) \\ = \sum_k \frac{\partial}{\partial x_k}\psi(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}(\mathbf{x}_0,t)} &\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t)\,.\end{aligned} \tag{4.3}$$

Making use of the chain rule, and employing the fact that $\mathbf{u}^{(L)} = \partial\mathbf{x}/\partial t$, we are led to

$$\sum_k \frac{\partial}{\partial t}\left(Y_k^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t)\right) = \frac{\partial}{\partial x_{0i}}\psi^{(L)}(\mathbf{x}_0, t). \tag{4.4}$$

Integrating with respect to t and using $(\partial x_k/\partial x_{0i})(\mathbf{x}_0, 0) = \partial x_{0k}/\partial x_{0i} = \delta_{ik}$, we end up with the following "generalized Weber transformation":

$$\sum_k Y_k^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t) = Y_i^{(L)}(\mathbf{x}_0, 0) + \frac{\partial}{\partial x_{0i}}\int_0^t \psi^{(L)}(\mathbf{x}_0, t')\,dt'\,. \tag{4.5}$$

Transformation to the Eulerian picture leads to

$$\begin{aligned}\sum_i \left(\sum_k Y_k^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_{0i}}x_k(\mathbf{x}_0, t)\right)\Big|_{\mathbf{x}_0=\mathbf{x}_0(\mathbf{x},t)} \frac{\partial}{\partial x_j}x_{0i}(\mathbf{x}, t) & \\ = \sum_i \left(Y_i^{(L)}(\mathbf{x}_0, 0) + \frac{\partial}{\partial x_{0i}}\int_0^t \psi^{(L)}(\mathbf{x}_0, t')\,dt'\right)\Big|_{\mathbf{x}_0=\mathbf{x}_0(\mathbf{x},t)} &\frac{\partial}{\partial x_j}x_{0i}(\mathbf{x}, t)\,.\end{aligned} \tag{4.6}$$

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Employing the chain rule, we finally get:

$$Y_j(\mathbf{x},t) = \sum_i Y_i^{(L)}(\mathbf{x}_0(\mathbf{x},t),0)\frac{\partial}{\partial x_j}x_{0i}(\mathbf{x},t) + \frac{\partial}{\partial x_j}\int_0^t \psi^{(L)}(\mathbf{x}_0(\mathbf{x},t),t')\,dt'\,. \tag{4.7}$$

Defining

$$\boldsymbol{\alpha}(\mathbf{x},t) = \mathbf{Y}^{(L)}(\mathbf{x}_0(\mathbf{x},t),0)\,, \qquad \phi(\mathbf{x},t) = \int_0^t \psi^{(L)}(\mathbf{x}_0(\mathbf{x},t),t')\,dt'\,, \tag{4.8}$$

we end up with the following decomposition of $\mathbf{Y}$:

$$\mathbf{Y}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t) + \sum_i \alpha_i(\mathbf{x},t)\nabla x_{0i}(\mathbf{x},t). \tag{4.9}$$

The time evolution equations of the fields α_i are simply

$$\frac{D}{Dt}\alpha_i(\mathbf{x},t) = 0, \tag{4.10}$$

due to the fact that $\alpha_i(\mathbf{x},t) = f_i(\mathbf{x}_0(\mathbf{x},t))$. Moreover, (4.8) implies

$$\frac{\partial}{\partial t}\phi^{(L)}(\mathbf{x}_0,t) = \psi^{(L)}(\mathbf{x}_0,t)\,. \tag{4.11}$$

In Eulerian coordinates, this corresponds to

$$\frac{D}{Dt}\phi(\mathbf{x},t) = \psi(\mathbf{x},t)\,. \tag{4.12}$$

Vice versa, the generalized Euler equation (4.1) can be rederived from (4.9) with (4.10) and (4.12). Using the obvious commutation relation

$$\nabla\frac{D}{Dt} - \frac{D}{Dt}\nabla = \sum_j (\nabla u_j)\frac{\partial}{\partial x_j} \tag{4.13}$$

we get

$$\begin{aligned}
\frac{D}{Dt}\mathbf{Y} &= \frac{D}{Dt}\nabla\phi + \sum_i \underbrace{\left(\frac{D}{Dt}\alpha_i\right)}_{=0}\nabla x_{0i} + \sum_i \alpha_i\frac{D}{Dt}\nabla x_{0i} \\
&= \nabla\underbrace{\left(\frac{D}{Dt}\phi\right)}_{=\psi} - \sum_j(\nabla u_j)\frac{\partial}{\partial x_j}\phi + \sum_i\alpha_i\nabla\underbrace{\left(\frac{D}{Dt}x_{0i}\right)}_{=0} - \sum_{i,j}\alpha_i(\nabla u_j)\frac{\partial}{\partial x_j}x_{0i} \\
&= \nabla\psi - \sum_j(\nabla u_j)Y_j\,.
\end{aligned} \tag{4.14}$$

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The results obtained so far can be summed up as follows:

Proposition 1. (Generalized Weber Transformation)

The validity of the generalized Euler equation $(D/Dt)\mathbf{Y} + \sum_j Y_j \nabla u_j = \nabla \psi$ gives rise to the following generalized Weber transformation:

$$\sum_k Y_k^{(L)}(\mathbf{x}_0, t)\frac{\partial}{\partial x_{0i}} x_k(\mathbf{x}_0, t) = Y_i^{(L)}(\mathbf{x}_0, 0) + \frac{\partial}{\partial x_{0i}} \int_0^t \psi^{(L)}(\mathbf{x}_0, t')\, dt' . \tag{4.15}$$

Proposition 2. (Generalized Lin Representation)

The validity of the generalized Euler equation $(D/Dt)\mathbf{Y} + \sum_j Y_j \nabla u_j = \nabla \psi$ is globally equivalent to the existence of the representation

$$\mathbf{Y} = \nabla \phi + \sum_i \alpha_i \nabla x_{0i} , \tag{4.16}$$

where α_i, x_{0i} and ϕ are solutions of the following time evolution equations:

$$\frac{D}{Dt}\alpha_i = 0 , \qquad \frac{D}{Dt} x_{0i} = 0 , \qquad \frac{D}{Dt}\phi = \psi . \tag{4.17}$$

With regard to the derivation of a generalized Clebsch transformation, we now make use of the fact that $\mathbf{Y}^{(L)}(\mathbf{x}_0, 0)$ can be locally represented in terms of Clebsch potentials:

$$Y_i^{(L)}(\mathbf{x}_0, 0) = \frac{\partial}{\partial x_{0i}} \eta_0(\mathbf{x}_0) + \alpha_0(\mathbf{x}_0)\frac{\partial}{\partial x_{0i}} \beta_0(\mathbf{x}_0) . \tag{4.18}$$

Thus, (4.7) implies

$$\begin{aligned} Y_j(\mathbf{x}, t) &= \sum_i Y_i^{(L)}(\mathbf{x}_0(\mathbf{x}, t), 0)\frac{\partial}{\partial x_j} x_{0i}(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\phi(\mathbf{x}, t) \\ &= \sum_i \left(\frac{\partial}{\partial x_{0i}}\eta_0(\mathbf{x}_0)\right)\bigg|_{\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x},t)} \frac{\partial}{\partial x_j} x_{0i}(\mathbf{x}, t) \\ &\quad + \sum_i \left(\alpha_0(\mathbf{x}_0)\frac{\partial}{\partial x_{0i}}\beta_0(\mathbf{x}_0)\right)\bigg|_{\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x},t)} \frac{\partial}{\partial x_j} x_{0i}(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\phi(\mathbf{x}, t) . \end{aligned} \tag{4.19}$$

Defining

$$\alpha(\mathbf{x}, t) = \alpha_0(\mathbf{x}_0(\mathbf{x}, t)) , \quad \beta(\mathbf{x}, t) = \beta_0(\mathbf{x}_0(\mathbf{x}, t)) , \quad \eta(\mathbf{x}, t) = \eta_0(\mathbf{x}_0(\mathbf{x}, t)) , \tag{4.20}$$

and using the chain rule, we get

$$Y_j(\mathbf{x}, t) = \frac{\partial}{\partial x_j}(\eta(\mathbf{x}, t) + \phi(\mathbf{x}, t)) + \alpha(\mathbf{x}, t)\frac{\partial}{\partial x_j}\beta(\mathbf{x}, t) . \tag{4.21}$$

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Putting $\tilde{\phi}(\mathbf{x},t) = \eta(\mathbf{x},t) + \phi(\mathbf{x},t)$, we end up with a Clebsch representation for $\mathbf{Y}$:

$$\mathbf{Y}(\mathbf{x},t) = \nabla\tilde{\phi}(\mathbf{x},t) + \alpha(\mathbf{x},t)\nabla\beta(\mathbf{x},t)\,, \tag{4.22}$$

where α, β, and η are solutions of

$$\frac{D}{Dt}\alpha = 0\,, \qquad \frac{D}{Dt}\beta = 0\,, \qquad \frac{D}{Dt}\eta = 0\,. \tag{4.23}$$

Moreover

$$\frac{D}{Dt}\tilde{\phi} = \frac{D}{Dt}\eta + \frac{D}{Dt}\phi = \frac{D}{Dt}\phi = \psi\,. \tag{4.24}$$

Vice versa, in analogy to (4.14) it can be shown that the generalized Euler equation is derivable from (4.22) together with the above time evolution equations for α, β, and $\tilde{\phi}$.

Again we summarize:

PROPOSITION 3. (Generalized Clebsch Transformation)

The validity of the generalized Euler equation $(D/Dt)\mathbf{Y} + \sum_j Y_j \nabla u_j = \nabla\psi$ is locally equivalent to the existence of the representation

$$\mathbf{Y} = \nabla\tilde{\phi} + \alpha\nabla\beta\,, \tag{4.25}$$

where α, β and $\tilde{\phi}$ are solutions of the following time evolution equations:

$$\frac{D}{Dt}\alpha = 0\,, \qquad \frac{D}{Dt}\beta = 0\,, \qquad \frac{D}{Dt}\tilde{\phi} = \psi\,. \tag{4.26}$$

We conclude the present section with the remark that due to

$$\begin{aligned}\frac{D}{Dt}\tilde{\phi} &= \frac{\partial}{\partial t}\tilde{\phi} + (\mathbf{u}\cdot\nabla)\tilde{\phi} = \frac{\partial}{\partial t}\tilde{\phi} + \mathbf{u}\cdot(\mathbf{Y} - \alpha\nabla\beta)\\ &= \frac{\partial}{\partial t}\tilde{\phi} + \mathbf{u}\cdot\mathbf{Y} - \alpha(\mathbf{u}\cdot\nabla)\beta = \frac{\partial}{\partial t}\tilde{\phi} + \mathbf{u}\cdot\mathbf{Y} - \alpha\left(\frac{D}{Dt}\beta - \frac{\partial}{\partial t}\beta\right)\\ &= \frac{\partial}{\partial t}\tilde{\phi} + \mathbf{u}\cdot\mathbf{Y} + \alpha\frac{\partial}{\partial t}\beta\,,\end{aligned} \tag{4.27}$$

the time evolution equation (4.24) may be replaced with the "generalized Bernoulli theorem"

$$\frac{\partial}{\partial t}\tilde{\phi} + \alpha\frac{\partial}{\partial t}\beta + \mathbf{u}\cdot\mathbf{Y} - \psi = 0\,. \tag{4.28}$$

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5. Examples

It remains to show that there indeed exists a sufficiently nontrivial list of hydrodynamical systems with Euler equations that can be cast into the generalized form (4.1).

The first example on this list to be mentioned is the non-barotropic ideal fluid. Here the energy density e, the temperature $T = \partial e/\partial s$, and the pressure $p = \varrho^2 \partial e/\partial \varrho$ are depending on both the mass density ϱ and the entropy density s. In addition to the Euler and the continuity equation we require the isentropy relation $Ds/Dt = 0$ to be valid.

We define the following quantity (sometimes called "thermasy" [13]):

$$\theta = \int_0^t T^{(L)}(\mathbf{x}_0(\mathbf{x},t),t')\,dt' . \tag{5.1}$$

It is a solution of $D\theta/Dt = T$ and one can show that the Euler equation

$$\frac{D}{Dt}\mathbf{u} = -\frac{\nabla p}{\varrho} - \nabla U \tag{5.2}$$

can be cast into the form of the generalized Euler equation (4.1) with

$$\mathbf{Y} = \mathbf{u} - \theta\nabla s\,, \qquad \psi = \frac{\mathbf{u}^2}{2} - e - \frac{p}{\varrho} - U\,. \tag{5.3}$$

If the forces exerted on the non-barotropic ideal fluid are due to an external electromagnetic field, then the respective Euler equation

$$\frac{D}{Dt}\mathbf{u} = -\frac{\nabla p}{\varrho} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B}\right) \tag{5.4}$$

can be cast into the form (4.1) with

$$\mathbf{Y} = \mathbf{u} + \frac{q}{mc}\mathbf{A} - \theta\nabla s\,, \qquad \psi = \frac{\mathbf{u}^2}{2} - e - \frac{p}{\varrho} - \frac{q}{m}\left(\varphi - \frac{1}{c}\mathbf{u}\cdot\mathbf{A}\right), \tag{5.5}$$

where φ and $\mathbf{A}$ denote the usual electromagnetic scalar and vector potentials.

Another example consists of the Euler equation for a non-barotropic ideal fluid in magnetohydrodynamics with infinite conductivity:

$$\frac{D}{Dt}\mathbf{u} = -\frac{\nabla p}{\varrho} + \frac{1}{4\pi\varrho}\left((\nabla\times\mathbf{B})\times\mathbf{B}\right) . \tag{5.6}$$

Introducing an auxiliary vector field $\mathbf{h}(\mathbf{x},t)$ *via*

$$h_j(\mathbf{x},t) = \frac{1}{4\pi}\sum_k \int_0^t B_k^{(L)}(\mathbf{x}_0(\mathbf{x},t),t')\frac{\partial}{\partial x_j}\Big(x_k(\mathbf{x}_0(\mathbf{x},t),t')\Big)dt' , \tag{5.7}$$

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Eq. (5.6) can be cast into the form (4.1) with

$$\mathbf{Y} = \mathbf{u} - \theta\nabla s - \frac{1}{\varrho}(\nabla \times \mathbf{h}) \times \mathbf{B}, \qquad \psi = \frac{\mathbf{u}^2}{2} - e - \frac{p}{\varrho}. \tag{5.8}$$

Here one has to make use of the fact that $\mathbf{h}(\mathbf{x}, t)$ is a solution of

$$\frac{\mathbf{B}}{4\pi} = \frac{\partial}{\partial t}\mathbf{h} - \mathbf{u} \times (\nabla \times \mathbf{h}) + \nabla(\mathbf{u}\cdot\mathbf{h}) \tag{5.9}$$

which implies

$$\frac{\nabla \times \mathbf{B}}{4\pi} = \frac{\partial}{\partial t}(\nabla \times \mathbf{h}) - \nabla \times [\mathbf{u} \times (\nabla \times \mathbf{h})]. \tag{5.10}$$

As a last example, we mention the following nonstationary extension of Darcy's law for the seepage of a barotropic fluid in a porous medium:

$$\frac{D}{Dt}\mathbf{u} = -\nabla P - \nabla U - \lambda\mathbf{u}. \tag{5.11}$$

Here, (4.1) can be achieved by putting

$$\mathbf{Y} = \mathbf{u}e^{\lambda t}, \qquad \psi = \left(\frac{\mathbf{u}^2}{2} - P - U\right)e^{\lambda t}. \tag{5.12}$$

For all these examples there exist Lagrangians for which – with the help of the generalized Lin representations – one can show that the Euler equation is always equivalently substituted within the respective Euler–Lagrange equations.

For the sake of completeness, we list up these Lagrangians but – due to the limited space – this has to be done without giving any further details:

(i) Non-barotropic ideal fluid [9]

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho e(\varrho, s) - \varrho U + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\theta\frac{D}{Dt}s - \varrho\boldsymbol{\alpha}\cdot\frac{D}{Dt}\mathbf{x}_0. \tag{5.13}$$

(ii) Non-barotropic ideal fluid in an external electromagnetic field

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho e(\varrho, s) - \frac{q}{m}\varrho\left(\varphi - \frac{1}{c}\mathbf{u}\cdot\mathbf{A}\right) + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\theta\frac{D}{Dt}s - \varrho\boldsymbol{\alpha}\cdot\frac{D}{Dt}\mathbf{x}_0. \tag{5.14}$$

(iii) Non-barotropic ideal fluid in magnetohydrodynamics with infinite conductivity [14]

$$\mathcal{L} = \frac{1}{2}\varrho\mathbf{u}^2 - \varrho e(\varrho, s) - \frac{\mathbf{B}^2}{8\pi} + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\theta\frac{D}{Dt}s - \varrho\boldsymbol{\alpha}\cdot\frac{D}{Dt}\mathbf{x}_0 - \mathbf{h}\cdot\left(\frac{\partial}{\partial t}\mathbf{B} - \nabla\times(\mathbf{u}\times\mathbf{B})\right) - \kappa\,\nabla\cdot\mathbf{B}. \tag{5.15}$$

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(iv) Barotropic fluid in a porous medium

$$\mathcal{L} = e^{\lambda t}\left\{\frac{1}{2}\varrho\mathbf{u}^2 - \varrho P(\varrho) + p(\varrho) - \varrho U\right\} + \phi\left(\frac{\partial}{\partial t}\varrho + \nabla\cdot(\varrho\mathbf{u})\right) - \varrho\boldsymbol{\alpha}\cdot\frac{D}{Dt}\mathbf{x}_0. \tag{5.16}$$

Making the replacement $\boldsymbol{\alpha}\cdot(D/Dt)\mathbf{x}_0 \to \alpha(D/Dt)\beta$ in the above expressions, one is left with Lagrangians that are no longer global but still local solutions of the inverse problem of Lagrangian field theory. This can be proved with the aid of the generalized Clebsch transformations discussed above.

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