

## Application of modified homotopy perturbation method to nonlinear oscillations

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THE PURPOSE OF THIS PAPER is to apply a version of homotopy technique to nonlinear problems. The modified version of homotopy perturbation method is applied to derive highly accurate analytical expressions for periodic solutions or for approximate formulas of frequency. In contrast with the traditional perturbation methods, the proposed method does not require any small parameter in the equation. The proposed algorithm avoids the complexity provided by other numerical approaches. The analysis is accompanied by three numerical examples. The results prove that this method is very effective and simple.

**Key words:** homotopy technique, strongly nonlinear oscillations, frequency, perturbation technique.

### 1. Introduction

THERE EXISTS A WIDE BODY of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies, called the perturbation methods. But almost all perturbation methods are based on small parameters so that the approximate solutions can be expanded in series of small parameters. Its basic idea is to transform, by means of small parameters, a nonlinear problem of an infinite number of linear subproblems into an infinite number of simpler ones. The small parameter determines not only the accuracy of the perturbation approximations but also the validity of the perturbation method.

There exist some analytical approaches, such as the harmonic balance method [1], the Krylov–Bogolyubov–Mitropolsky method [2], weighted linearization method [3], modified Lindstedt–Poincaré method [5], ADOMIAN decomposition method [6], artificial parameter method [7], homotopy perturbation method and so on.

In science and engineering, there exist many nonlinear problems, which do not contain any small parameters, especially those with strong nonlinearity. Thus it is necessary to develop and improve some nonlinear analytical approximations even for large parameters.

In 1992, LIAO [12] has proposed a new analytical method called the Homotopy Analysis Method, which introduces an embedding parameter to construct a homotopy and then analyzes it by means of the Taylor formula. Subsequently, by means of a linear property of homotopy, one can transform a nonlinear problem into an infinite number of linear subproblems, whether the nonlinear problem contains small parameters or not. Therefore, unlike the perturbation method, this method is independent of small parameters and can overcome the restrictions of the perturbation methods. LIAO and CHWANG [11] successfully applied the Homotopy Analysis Method to solve some simple nonlinear problems. Homotopy is an important part of differential topology so that it has a solid mathematical base [13]. The homotopy perturbation method provides a universal technique to introduce a perturbative parameter.

In this paper, we will apply the homotopy technique in a completely different way as in Refs. [7–12]. Let us further consider the damped, forced oscillation of a nonconservative nonlinear system governed by the equation

$$(1.1) \quad \ddot{u}(t) + \omega_0^2 u(t) = F(\Omega t, u, \dot{u}, \dots, {}^{(i)}\dot{u})$$

where  $u(t)$  is a dimensionless variable,  $\dot{u}(t) = \frac{du}{dt}$ ;  ${}^{(i)}\dot{u}(t) = \frac{d^i u(t)}{dt^i}$ ,  $i = 1, 2, \dots$ ,  $F$  is a nonlinear function, with the period  $T$  in the first variable,  $\omega_0$  and  $\Omega$  are constants.

Our purpose in this paper is to present the modified homotopy perturbation method for solving nonlinear problems, some ideas and improvements which point towards new and interesting applications of this method. We apply the modified homotopy perturbation method to three examples with large and small parameters. The periodic solutions obtained by this method are valid not only for small parameters, but also for very large parameters. This method sometimes leads to the results according to the standard Lindstedt–Poincaré method or the harmonic balance method.

## 2. Basic ideas of the modified homotopy perturbation method

We construct a one-parameter family of equations

$$(2.1) \quad \frac{\partial^2 U(t; p)}{\partial t^2} + \Lambda(p)U(t; p) = pF\left(\Omega t, U(t; p), \frac{\partial U(t; p)}{\partial t}, \dots, \frac{\partial^i U(t; p)}{\partial t^i}\right),$$

$$t \in [0, \infty), \quad p \in [0, 1], \quad i = 1, 2, \dots$$

where  $p$  is considered as an expanding parameter and  $U(t; p)$  is an analytical function of both  $t$  and  $p$ .

At  $p = 0$ , we have obviously  $U(t; 0) = u_0(t)$  and  $u_0(t)$  is an initial approximation of Eq. (1.1) which not necessarily satisfies the boundary conditions. At  $p = 1$ , Eq. (2.1) is exactly the same as Eq. (1.1), respectively, so that  $U(t; 1) = u(t)$  and  $u(t)$  is exactly the solution that we want to know. As the expanding parameter  $p$  varies from zero to one,  $U(t; p)$  varies continuously from  $u_0(t)$  to  $u(t)$  and  $\Lambda(p)$  varies from  $\Lambda(0) = \omega^2$  to  $\Lambda(1) = \omega_0^2$ , where  $\omega$  is the angular frequency of the system (1.1). This kind of continuous variations is called deformation in topology [10, 11]. The continuous deformations of  $U(t; p)$  and  $\Lambda(p)$  are completely governed by Eq. (2.1).

Here, we emphasize that neither small nor large parameters are necessary in constructing the zeroth-order deformation Eq. (2.1). In fact, whether or not Eq. (1.1) contains small or large parameters, it is not important at all for the validity of the modified homotopy method, because the only assumptions made in Eq. (1.1) are that  $F$  should be analytical and with the period  $T$  in the first variable.

Suppose that  $U(t; p)$  and  $\Lambda(p)$  have derivatives with respect to the expanding variable  $p$  evaluated at  $p = 0$ :

$$(2.2) \quad \left. \frac{\partial^j U(t; p)}{\partial p^j} \right|_{p=0} = u_0^{[j]}(t), \quad \left. \frac{\partial^j \Lambda(p)}{\partial p^j} \right|_{p=0} = \Lambda_0^{[j]}, \quad j \geq 1$$

which are called the  $j$  th-order deformation derivatives. By Taylor's formula, we have:

$$(2.3) \quad U(t; p) = u_0(t) + \sum_{j \geq 1} \frac{u_0^{[j]}(t)}{j!} p^j$$

and

$$(2.4) \quad \Lambda(p) = \Lambda(0) + \sum_{j \geq 1} \frac{\Lambda_0^{[j]}}{j!} p^j.$$

Setting  $p = 1$ , we obtain

$$(2.5) \quad u(t) = u_0(t) + \sum_{j \geq 1} \frac{u_0^{[j]}(t)}{j!}$$

and

$$(2.6) \quad \omega_0^2 = \omega^2 + \sum_{j \geq 1} \frac{\Lambda_0^{[j]}}{j!}$$

provided that the radii of convergence of series (2.3) and (2.4) are not less than 1. Note that (2.5) gives a relation between the initial approximation  $u_0(t)$  and solution  $u(t)$ ; meanwhile, (2.6) provides a link between the initial approximation  $\Lambda(0) = \omega^2$  and the square of the frequency  $\omega_0$ . Now, the key to the problem becomes: how to solve these  $j$  th-order deformation derivatives  $u_0^{[j]}(t)$  and  $\Lambda_0^{[j]}$ , ( $j \geq 1$ ). For this purpose, we must first of all give equations governing  $u_0(t)$ ,  $u_0^{[j]}(t)$  and  $\Lambda_0^{[j]}$ , ( $j \geq 1$ ).

Setting  $p = 0$  we obtain the equation:

$$(2.7) \quad \ddot{u}_0(t) + \omega^2 u_0(t) = 0.$$

Differentiating Eq. (2.1) with respect to  $p$  and setting  $p = 0$ , we have:

$$(2.8) \quad \ddot{u}_0^{[1]}(t) + \omega^2 u_0^{[1]}(t) = F(\Omega t, u_0(t), \dot{u}_0(t), \dots, u_0^{(i)}(t)) - \Lambda_0^{[1]} u_0(t)$$

where  $u_0(t)$  is given by Eq. (2.7). Avoiding the secular term, we obtain  $\Lambda_0^{[1]}$  and the relationship between the constants of integration from Eq. (2.7). We call Eq. (2.8) the first-order deformation equation. In the same way, we can obtain all of the  $j$  th-order deformation equations governing  $u_0^{[j]}(t)$  ( $j \geq 2$ ), which are similar in form to Eq. (2.8) except for the inhomogeneous terms. For example, for  $j = 2$ , we obtain the second-order deformation equation:

$$(2.9) \quad \ddot{u}_0^{[2]}(t) + \Lambda(0) u_0^{[2]}(t) = 2F^{[1]}(\Omega t, u_0(t), \dot{u}_0(t), \dots, u_0^{(i)}(t)) - 2\Lambda_0^{[1]} u_0^{[1]}(t) - \Lambda_0^{[2]} u_0(t)$$

where

$$(2.10) \quad F^{[1]}(\Omega t, u_0, \dots, u_0^{(i)}) = \left. \frac{dF(\Omega t, u, \dot{u}, \dots, u^{(i)})}{dp} \right|_{p=0} \\ = \left. \frac{\partial F}{\partial u} \right|_{p=0} u_0^{[1]}(t) + \sum_{k=1}^i \left. \frac{\partial F}{\partial u^{(k)}} \right|_{p=0} u_0^{(k)[1]}(t)$$

with  $u_0^{[1]}(t)$  given by Eq. (2.8), and  $\Lambda_0^{[2]}$  can be determined as above.

Let us emphasize that the first-order deformation Eq. (2.8) is linear with respect to the first-order deformation derivative  $u_0^{[1]}(t)$ . In fact, every  $j$  th-order deformation equation is linear with respect to the corresponding  $j$  th-order deformation derivative ( $j \geq 1$ ). It means that every term  $u_0^{[j]}(t)$  ( $j \geq 1$ ) in (2.5) is

governed by a linear equation. Therefore, by (2.5), the original nonlinear problem governed by Eq. (1.1) can be transformed to an infinite number of linear sub-problems about the  $j$  th-order deformation derivatives  $u_0^{[j]}(t)$  ( $j \geq 1$ ). By means of the modified homotopy perturbation method, we also accomplish this kind of transformation but without using the small parameter assumption. Thus, this method is in principle different from the perturbation method. Details will be discussed below.

Another case to be analysed is that there is a real parameter  $\varepsilon$  (small or large) such as  $F\left(\Omega t, u, \dot{u}, \dots, \overset{(i)}{u}\right) = \varepsilon f\left(\Omega t, u, \dot{u}, \dots, \overset{(i)}{u}\right)$ . Equation (1.1) becomes:

$$(2.11) \quad \ddot{u}(t) + \omega^2 u(t) = \varepsilon f\left(\Omega t, u, \dot{u}, \dots, \overset{(i)}{u}\right).$$

With the notations:

$$(2.12) \quad u_0^{[j]} = \varepsilon^j u_0^{(j)}; \quad \Lambda_0^{[j]} = \varepsilon^j \Lambda_0^{(j)}; \quad j \geq 1$$

Eqs. (2.5), (2.6), (2.7), (2.8) and (2.9) are respectively

$$(2.13) \quad u(t) = u_0 + \sum_{j \geq 1} \frac{\varepsilon^j u_0^{(j)}(t)}{j!},$$

$$(2.14) \quad \omega_0^2 = \omega^2 + \sum_{j \geq 1} \frac{\varepsilon^j \Lambda_0^{(j)}}{j!},$$

$$(2.15) \quad \ddot{u}_0(t) + \omega^2 u_0(t) = 0,$$

$$(2.16) \quad \ddot{u}_0^{(1)}(t) + \omega^2 u_0^{(1)}(t) = f\left(\Omega t, u_0, \dot{u}_0, \dots, \overset{(i)}{u}_0\right) - \Lambda_0^{(1)} u_0(t),$$

$$(2.17) \quad \ddot{u}_0^{(2)}(t) + \omega^2 u_0^{(2)}(t) = 2f^{(1)}\left(\Omega t, u_0, \dot{u}_0, \dots, \overset{(i)}{u}_0\right) - 2\Lambda_0^{(1)}(t) - \Lambda_0^{(2)} u_0(t),$$

where

$$(2.18) \quad f^{(1)}\left(\Omega t, u_0, \dot{u}_0, \dots, \overset{(i)}{u}_0\right) = \left. \frac{df(\Omega t, u, \dot{u}, \dots, \overset{(i)}{u})}{dp} \right|_{p=0} \\ = \left. \frac{\partial f}{\partial u} \right|_{p=0} u_0^{(i)}(t) + \sum_{k=1}^i \left. \frac{\partial f}{\partial \overset{(k)}{u}} \right|_{p=0} \overset{(k)(1)}{u}_0(t), \dots$$

### 3. Some applications

We illustrate the basic evaluation procedure of the proposed method by following three examples:

#### 3.1. Example 1

We consider the differential equation [10]:

$$(3.1) \quad \ddot{u} + \frac{u}{1 + \varepsilon u^2} = 0, \quad \varepsilon \in (0, \infty)$$

with initial conditions:

$$(3.2) \quad u(0) = A \quad \text{and} \quad \dot{u}(0) = 0.$$

We rewrite Eq. (3.1) in the form

$$(3.3) \quad \ddot{u} + u = -\varepsilon u^2 \ddot{u}.$$

By the formula (2.15), we have

$$(3.4) \quad u_0(t) = A \cos \omega t.$$

Substituting (3.4) into (2.16), results in:

$$(3.5) \quad \ddot{u}_0^{(1)} + \omega^2 u_0^{(1)} = \varepsilon \omega^2 A^3 \cos^3 \omega t - \Lambda_0^{(1)} A \cos \omega t$$

or

$$(3.6) \quad \ddot{u}_0^{(1)} + \omega^2 u_0^{(1)} = A \cos \omega t \left( \frac{3\omega^2}{4} A^2 - \Lambda_0^{(1)} \right) + \frac{\omega^2 A^3}{4} \cos 3\omega t.$$

Avoiding the presence of a secular term in Eq. (3.6), needs:

$$(3.7) \quad \Lambda_0^{(1)} = \frac{3\omega^2}{4} A^2.$$

Considering the initial conditions  $u_0^{(1)} = 0$  and  $\dot{u}_0^{(1)}(0) = 0$ , we obtain the solution of Eq. (3.6), which reads:

$$(3.8) \quad u_0^{(1)}(t) = \frac{A^3}{32} (\cos \omega t - \cos 3\omega t).$$

Substitution of Eqs. (3.4), (3.7), and (3.8) into Eq. (2.17) yields:

$$(3.9) \quad \ddot{u}_0^{(2)} + \omega^2 u_0^{(2)} = A \cos \omega t \left( \frac{3}{128} \omega^2 A^4 - \Lambda_0^{(2)} \right) - \frac{13}{128} A^4 \omega^2 \cos 3\omega t - \frac{11}{128} A^4 \omega^2 \cos 5\omega t.$$

The elimination of secular term requires:

$$(3.10) \quad A_0^{(2)} = \frac{3}{128} \omega^2 A^4.$$

Solving Eq. (3.9) with the initial conditions  $u_0^{(2)} = 0$  and  $\dot{u}_0^{(2)}(0) = 0$ , we obtain

$$(3.11) \quad u_0^{(2)}(t) = \frac{13A^4}{1024} (\cos 3\omega t - \cos \omega t) + \frac{11A^4}{3072} (\cos 5\omega t - \cos \omega t).$$

Substituting  $A_0^{(1)}$  and  $A_0^{(2)}$  into (2.14), ( $\omega_0 = 1$ ), we have:

$$(3.12) \quad 1 = \omega^2 + \frac{3}{4} \varepsilon \omega^2 A^2 + \frac{3}{256} \varepsilon^2 \omega^2 A^4 + O(\varepsilon^3).$$

From Eq. (3.12) we obtain

$$(3.13) \quad \omega^2 = \frac{1}{1 + \frac{3}{4} \varepsilon A^2 + \frac{3}{256} \varepsilon^2 A^4}.$$

The approximation period obtained from Eq. (3.13) is

$$(3.14) \quad T_{\text{approx}} = 2\pi \left( 1 + \frac{3}{4} \varepsilon A^2 + \frac{3}{256} \varepsilon^2 A^4 \right)^{-1/2}.$$

The formula (3.14) works well for small  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) but breaks down quickly when  $\varepsilon$  becomes large. Here, we wish to develop uniformly valid expansions for  $\omega^2$  and  $u(t)$  for large values of  $\varepsilon$ , using a newly defined expansion parameter  $\eta(\varepsilon, A)$  from (3.12) as follows [5]:

$$(3.15) \quad \eta(\varepsilon, A) = \frac{\frac{3}{4} \varepsilon A^2}{1 + \frac{3}{4} \varepsilon A^2}.$$

This relation is quickly convergent regardless of the magnitude of  $\varepsilon A^2$ , since  $\eta < 1$  for all  $\varepsilon A^2$ . In terms of  $\eta$ , the original parameter  $\varepsilon$  is given by

$$(3.16) \quad \varepsilon = \frac{\eta}{\frac{3}{4} A^2 (1 - \eta)}.$$

Equation (3.3) can be rewritten as

$$(3.17) \quad \ddot{u} + u = \eta \left( \ddot{u} + u - \frac{4u^2 \ddot{u}}{3A^2} \right).$$

Equation (2.15), (2.16) and (2.17) are respectively:

$$(3.18) \quad \ddot{u}_0 + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad \dot{u}_0(0) = 0,$$

$$(3.19) \quad \begin{aligned} \ddot{u}_0^{(1)} + \omega^2 u_0^{(1)} &= -\Lambda_0^{(1)} u_0 + \ddot{u}_0 + u_0 - \frac{4u_0^2 \ddot{u}}{3A^2}, \\ u_0^{(1)}(0) &= 0, \\ \dot{u}_0^{(1)}(0) &= 0, \end{aligned}$$

$$(3.20) \quad \begin{aligned} \ddot{u}_0^{(2)} + \omega^2 u_0^{(2)} &= -\Lambda_0^{(2)} u_0 - 2\Lambda_0^{(1)} u_0^{(1)} + 2\ddot{u}_0^{(1)} + 2u_0^{(1)} \\ &\quad - \frac{8 \left( u_0^2 \ddot{u}_0^{(1)} + 2u_0 \ddot{u}_0 u_0^{(1)} \right)}{3A^2}. \end{aligned}$$

Equation (3.18) has the solution

$$(3.21) \quad u_0(t) = A \cos \omega t.$$

Substituting (3.21) into (3.19), by the simple manipulation, we have

$$(3.22) \quad \ddot{u}_0^{(1)} + \omega^2 u_0^{(1)} = A \cos \omega t \left( 1 - \Lambda_0^{(1)} \right) + \frac{A\omega^2}{3} \cos 3\omega t.$$

In order to ensure that no secular term appears in Eq. (3.22), the resonance must be avoided. To do so, coefficient of  $\cos \omega t$  must be zero, i.e.

$$(3.23) \quad \Lambda_0^{(1)} = 1.$$

Assuming the initial conditions  $u_0^{(1)} = 0$  and  $\dot{u}_0^{(1)} = 0$  in (3.22), we obtain

$$(3.24) \quad u_0^{(1)}(t) = \frac{A}{24} (\cos \omega t - \cos 3\omega t).$$

The substitution of (3.21), (3.23) and (3.24) into (3.20) yields:

$$(3.25) \quad \begin{aligned} \ddot{u}_0^{(2)} + \omega^2 u_0^{(2)} &= A \cos \omega t \left( -\Lambda_0^{(2)} - \frac{5\omega^2}{36} \right) \\ &\quad + \frac{2A\omega^2}{9} \cos 3\omega t - \frac{11\omega^2 A}{36} \cos 5\omega t. \end{aligned}$$



Avoiding the presence of a secular term in Eq. (3.25), we obtain:

$$(3.26) \quad A_0^{(2)} = -\frac{5\omega^2}{36}.$$

In the initial conditions  $u_0^{(2)}(0) = 0$  and  $\dot{u}_0^{(2)}(0) = 0$ , we have:

$$(3.27) \quad u_0^{(2)}(t) = \frac{A}{36} (\cos \omega t - \cos 3\omega t) + \frac{11A}{864} (\cos 5\omega t - \cos \omega t).$$

Substituting (3.23) and (3.26) into equation

$$(3.28) \quad 1 = \omega^2 + \eta A_0^{(1)} + \frac{\eta}{2} A_0^{(2)} + 0(\eta^3).$$

we obtain:

$$(3.29) \quad \omega^2 = \frac{1 - \eta}{1 - \frac{5}{72}\eta^2}.$$

Substituting (3.15) into (3.29) we have

$$(3.30) \quad \omega^2 = \frac{96\varepsilon A^2 + 128}{67\varepsilon^2 A^4 + 192\varepsilon A^2 + 128}.$$

The approximation period obtained from (3.30) is

$$(3.31) \quad T_{\text{approx}} = 2\pi \left( \frac{67\varepsilon^2 A^4 + 192\varepsilon A^2 + 128}{96\varepsilon A^2 + 128} \right)^{1/2}.$$

To illustrate the remarkable accuracy of the obtained results, we compare the approximate period (3.31) with the exact one [10]:

$$(3.32) \quad T_{\text{ex}} = 4\sqrt{\varepsilon} \int_0^A \frac{du}{\sqrt{\ln(1 + \varepsilon A^2) - \ln(1 + \varepsilon u^2)}}.$$

In case  $\varepsilon A^2 \rightarrow \infty$ , we have

$$(3.33) \quad \lim_{\varepsilon A^2 \rightarrow \infty} \frac{T_{\text{ex}}}{T_{\text{approx}}} = \frac{2\sqrt{2\pi\varepsilon}A}{2\pi\sqrt{\frac{67}{96}\varepsilon A^2}} \approx 0.955076.$$

Therefore, for any values of  $\varepsilon$ , it can be easily proved that the maximal relative error is less than 4.5%.

REMARK 1. We compare our procedure with the homotopy perturbation method. HE [9] constructed for Eq. (3.1) a homotopy which satisfies:

$$(3.34) \quad (1-p)[L(v) - L(u_0)] + p[(1 + \varepsilon v^2)\ddot{v} + v] = 0$$

where  $L(v) = \ddot{v} + v$ . The initial approximation of Eq. (3.3) is assumed in the form:

$$(3.35) \quad u_0(t) = A \cos \alpha t$$

where  $\alpha(\varepsilon)$  is a non-zero unknown constant with  $\alpha(0) = 1$ . The basic assumption is that the solution of (3.3) can be written as a power series in  $p$ :

$$(3.36) \quad v = v_0 + pv_1 + p^2v_2 + \dots$$

Setting  $p = 1$  results in the approximate solution of Eq. (3.3)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

The series (3.36) may converge in the whole solution domain as  $p$  tends to one. Substituting Eq. (3.36) into Eq. (3.34), and equating the terms with the identical powers of  $p$ , we have:

$$(3.37) \quad L(v_0) - L(u_0) = 0, \quad v_0(0) = A, \quad \dot{v}(0) = 0$$

$$(3.38) \quad \begin{aligned} L(v_1) - L(v_0) + L(u_0) + (1 + \varepsilon v_0^2) \ddot{v}_0 + v_0 &= 0; \\ \dot{v}_1(0) &= 0; \quad v_1(0) = 0. \end{aligned}$$

Setting  $v_0 = u_0 = A \cos \alpha t$ , the unknown  $\alpha$  can be determined by the Galerkin method:

$$(3.39) \quad \int_0^{\pi/\alpha} \sin \alpha t [(1 + \varepsilon u_0^2) \ddot{u}_0 + u_0] dt = 0.$$

The unknown  $\alpha$  therefore can be identified:

$$(3.40) \quad \alpha = \left(1 + \frac{3}{4} \varepsilon A^2\right)^{-1/2}.$$

As a result, from Eq. (3.38) we obtain:

$$(3.41) \quad \ddot{v}_1 + v_1 - \alpha^2 \frac{\varepsilon A^3}{4} \cos 3\alpha t = 0$$

with the solution

$$(3.42) \quad v_1(t) = -\frac{\alpha^2 \varepsilon A^3}{4(9\alpha^2 - 1)}(\cos 3\alpha t - \cos t).$$

If, for example, the first-order approximation is sufficient, then we have

$$(3.43) \quad u(t) = \lim_{p \rightarrow 1} v(t) = v_0(t) + v_1(t) = A \cos \alpha t - \frac{\alpha^2 \varepsilon A^3}{4(9\alpha^2 - 1)}(\cos 3\alpha t - \cos t)$$

with  $\alpha$  defined as in Eq. (3.40).

The period of the solution can be expressed as follows:

$$(3.44) \quad T = 2\pi \left(1 + \frac{3}{4}\varepsilon A^2\right)^{1/2}.$$

The first order approximation of Eq. (3.1) or (3.3) by our procedure is obtained from (2.13), (3.4) and (3.8):

$$(3.45) \quad u^*(t) = A \cos \omega t + \frac{A^3}{32}(\cos \omega t - \cos 3\omega t)$$

and  $\omega$  is obtained from (2.14) ( $\omega_0 = 1$ ):

$$(3.46) \quad \omega^* = \left(1 - \frac{3}{4}\varepsilon A^2\right)^{1/2}$$

while the period of the solution is given by the expression:

$$(3.47) \quad T^* = 2\pi \left(1 - \frac{3}{4}\varepsilon A^2\right)^{-1/2}$$

In our procedure, the term  $u_0(t)$  results from Eq. (2.15), while with homotopy perturbation method this term is supposed to be in the form (3.35). In both the methods, the results are closed for  $\varepsilon$  small. The methods differ by the choice of homotopies, of the frequencies and of the solutions.

### 3.2. Example 2

Now, we consider the motion of a nonlinear Mathieu oscillator, in one spatial dimension [14]

$$(3.48) \quad \ddot{u} + \omega_0^2 u = \varepsilon(u^3 - u) \cos 2t$$

with the initial conditions:  $u(0) = a$  and  $\dot{u}(0) = 0$  and for the case  $\omega_0 \approx 1$ . We consider  $\Lambda(0) = \omega^2 = 1$  and thus Eq. (2.15) becomes:

$$(3.49) \quad \ddot{u}_0(t) + u_0(t) = 0.$$

Solving Eq. (3.49) with the recalled initial conditions, we obtain:

$$(3.50) \quad u_0(t) = a \cos t.$$

Substituting (3.50) into (2.16), we obtain

$$(3.51) \quad \ddot{u}_0^{(1)}(t) + u_0^{(1)}(t) = \left[ \frac{1}{2} (a^3 - a) - \Lambda_0^{(1)} a \right] \cos t \\ + \frac{1}{8} (3a^3 - 4a) \cos 3t + \frac{1}{8} a^3 \cos 5t.$$

Avoiding the presence of a secular term requires:

$$(3.52) \quad \Lambda_0^{(1)} = \frac{1}{2} (a^2 - 1).$$

Considering the initial conditions  $u_0^{(1)}(0) = 0$  and  $\dot{u}_0^{(1)}(0) = 0$ , we obtain the solution of Eq. (3.51), which reads:

$$(3.53) \quad u_0^{(1)}(t) = \frac{1}{64} (4a - 3a^3) (\cos 3t - \cos t) + \frac{1}{192} a^3 (\cos t - \cos 5t).$$

Substituting (3.53) into (2.13) and (3.52) into (2.14) respectively, we have:

$$(3.54) \quad u(t) = a \cos t + \varepsilon \left[ \frac{1}{64} (4a - 3a^3) (\cos 3t - \cos t) \right. \\ \left. + \frac{1}{192} a^3 (\cos t - \cos 5t) \right] + 0(\varepsilon^2),$$

$$(3.55) \quad \omega_0^2 = 1 + \frac{\varepsilon}{2} (a^2 - 1) + 0(\varepsilon^2).$$

The formula (3.55) works well for small  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) but breaks down quickly when  $\varepsilon$  becomes large.

We use a newly defined expansion parameter  $\eta(\varepsilon, a)$

$$(3.56) \quad \eta(\varepsilon, a) = \frac{\varepsilon(a^2 - 1)}{2 + \varepsilon(a^2 - 1)}.$$

This relation is quickly convergent regardless of the magnitude of  $\varepsilon(a^2 - 1)$ , since  $\eta < 1$  for all  $\varepsilon(a^2 - 1)$ . In terms of  $\eta$ , the original parameter  $\varepsilon$  is given by

$$(3.57) \quad \varepsilon = \frac{2\eta}{(1 - \eta)(a^2 - 1)}.$$

Equation (3.48) can be rewritten as (strongly nonlinear Mathieu oscillator)

$$(3.58) \quad \ddot{u} + \omega^2 u = \eta \left[ \frac{2}{a^2 - 1} (u^3 - u) \cos 2t + \ddot{u} + \omega^2 u \right]$$

and the initial conditions are  $u(0) = a$  and  $\dot{u}(0) = 0$ .

By the same manipulation as the above example, we have:

$$(3.59) \quad u(t) = a \cos t + \eta \left[ \frac{a(4 - 3a^2)}{32(a^2 - 1)} (\cos 3t - \cos t) - \frac{a^5}{96(a^2 - 1)} (\cos 5t - \cos t) \right] \\ + \frac{\eta^2}{2} \left[ \frac{59a^5 - 138a^3 + 80a}{256(a^2 - 1)^2} (\cos t - \cos 3t) + \frac{35a^4 - 14a^3 - 24a}{2304(a^2 - 1)^2} (\cos t - \cos 5t) \right. \\ \left. + \frac{33a^5 - 40a^3}{3456(a^2 - 1)^2} (\cos 7t - \cos t) + \frac{a^5}{5120(a^2 - 1)^2} (\cos 9t - \cos t) \right] + 0(\eta^3),$$

$$(3.60) \quad \omega^2 = 1 + \frac{1}{2} \varepsilon (a^2 - 1) - \frac{\varepsilon^2 (a^4 - 12a^2 + 12)}{192[2 + \varepsilon(a^2 - 1)]}.$$

Comparing Eq. (3.60) with the numerical results for Eq. (3.48), we find a good agreement for the case  $a = 11/10$  (see Table 1).

**Table 1.**

$\varepsilon$	$\eta$	$\omega^2$ Eq. (3.60)	$\omega^2$ (numerical)
0.1	0.010391	1.0105272	1.01052
0.5	0.049881	1.0526625	1.05312
1	0.095022	1.1074875	1.10851
2	0.173553	1.2190548	1.22001
3	0.239543	1.3338179	1.35189
4	0.295774	1.4509812	1.48132

From Table 1, we note that the results obtained with our method work well and the frequency  $\omega^2$  will not deviate significantly from its leading order value 1. The analytical research result shows the applicability of the modified version to this kind of strongly nonlinear oscillator.

### 3.3. Example 3

In the last example, let us consider the well-known Duffing equation [1, 10, 11]:

$$(3.61) \quad \ddot{u} + u = -\varepsilon u^3$$

with the initial conditions  $u(0) = A$  and  $\dot{u}(0) = 0$ .

We obtain

$$(3.62) \quad \omega = \sqrt{\frac{1}{2} \left( 1 + \frac{3}{4} \varepsilon A^2 \right) + \frac{1}{2} \sqrt{1 + \frac{3}{2} \varepsilon A^2 + \frac{15}{32} \varepsilon^2 A^4}},$$

$$(3.63) \quad u(t) = A \cos \omega t + \frac{\varepsilon A^3}{32 \omega^2} (\cos 3\omega t - \cos \omega t) + \frac{\varepsilon^2 A^5}{1024 \omega^2} (\cos 5\omega t - \cos \omega t).$$

The exact frequency of the periodic motion of the Duffing equation is given by [10]:

$$(3.64) \quad \omega_{\text{ex}} = \frac{\pi \sqrt{1 + \varepsilon A^2}}{2} \left( \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m \sin^2 x}} \right)^{-1}$$

where  $m = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}$ . For comparison, the exact frequency obtained by integrating Eq. (3.64) and the approximate frequency computed by Eq. (3.62) are listed in Table 2. We also have

$$(3.65) \quad \lim_{\varepsilon A^2 \rightarrow \infty} \frac{\omega}{\omega_{\text{ex}}} = \frac{\sqrt{6 + \sqrt{30}}}{2\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5 \sin^2 x}} = 0.999699.$$

Note that the accuracy of (3.62) is not strongly dependent upon the values of  $\varepsilon A^2$  because they are uniformly valid for any possible values of  $\varepsilon A^2$ . Equation (3.65) shows that formula (3.62) can give an excellent approximate frequency for both small and large values of the oscillation amplitude. Therefore, for any value of  $\varepsilon > 0$  it can be easily proved that the maximal relative error of the frequency (3.62) is less than 0,3%. Without any cumbersome procedure, we can readily obtain the third or higher order approximates with high accuracy.

**Table 2.**

$\varepsilon A^2$	$\omega$ Eq. (3.62)	$\omega$ Eq. (3.64)
0.2	1.07200	1.07200
0.4	1.138906	1.13891
0.6	1.201731	1.20173
0.8	1.2611777	1.26118
1	1.3177644	1.31778
2	1.5690506	1.56911
5	2.1501774	2.15042
10	2.86612768	2.86664
100	8.5310997	8.53359
1000	26.802504	26.8107
10000	84.7013205	84.7245

#### 4. Conclusions

In this paper, we have studied analytically periodic solutions of strongly nonlinear oscillators. The modified homotopy perturbation method have been proved to be effective and have some distinct advantages over usual approximate methods in that the approximate (or even exact) solutions obtained in the present paper are valid not only for weakly nonlinear equations, but also for strongly nonlinear ones. In particular, it would be desirable to determine easier ways of constructing trial functions for some complex nonlinear problems. The results of this work presented here, not only demonstrate the applicability of the modified version of the homotopy perturbation method to strongly nonlinear oscillators, but also underline the importance of the periodic solutions in gaining a better understanding of physically relevant models. Convergence and error study for the above mentioned examples is a further need and it is clear that many other modifications can be made. This paper shows one step in the attempt to develop a new nonlinear analytical technique in absence of small parameters.

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