Brief Note

Simplest-expression integrals involving beam-column eigenmodes

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NOVEL INTEGRALS, IN THEIR SIMPLEST FORM, involving beam-column eigenfunctions and derivatives are presented. All the integrals arise in computational stability analysis of frames. It is stressed that simpler results improve the accuracy of Rayleigh–Ritz-based approximate methods for buckling analysis of distributed structures.

Key words: simplest-expression integrals, beam-column eigenfunctions, Rayleigh–Ritz method, stability, buckling.

1. Introduction

FREE VIBRATION AND STABILITY OF CONTINUOUS STRUCTURES such as beams, plates, shells, frames and other multibody structures composed of the ones just mentioned above, can be carried out by means of the classical Rayleigh–Ritz method and its generalization for composite systems – the substructure synthesis method [1–3]. In the analyses, admissible functions such as Euler–Bernoulli beam eigenfunctions (free vibration and stability through the vibration method) and beam-column eigenfunctions (stability through the static method) have the advantage, compared with other functions, of being mechanically related to the foregoing structures, which implies excellent convergence characteristics of the methodologies [2–4], and of being orthogonal. In fact, these were applied successfully in analyzing those structures throughout the last century [3, 5–10].

Nevertheless, the application of beam eigenfunctions has not been widely utilized because these have the drawback of being troublesome to work with, as they are complicated and involve hyperbolic and trigonometric functions. In order to avoid the use of beam functions, some researchers utilized non-orthogonal polynomials [11] and orthogonal polynomials [12], while others developed the finite element method which, as far as mechanics is concerned, represents the Rayleigh–Ritz method. The finite element method is not the most indicated one for beams, plates, shells and multibody structures with simple geometry; indeed, the superior convergence characteristics of basic Rayleigh–Ritz methodologies based on beam characteristic functions have been shown in these cases [2–4, 10, 13].

Consequently, if convergence speed is vital in analyzing this class of structures, the Rayleigh–Ritz method and the substructure synthesis method should be selected; thus, integrals involving beam eigenfunctions and derivatives must be negotiated. Moreover, because any of the Rayleigh–Ritz approaches is a numerical method itself, there is the strongest interest in reducing the number of computer operations in the algorithms, needless to say, in avoiding secondary routines as numerical integration. Reducing the number of computer operations is especially important because badly behaved hyperbolic functions and trigonometric functions are involved. Hence, the solution of the integrals must be simple.

Robert P. Felgar, in his not fully-published and sometimes disregarded work, presented a table with integrals involving Euler–Bernoulli beam eigenmodes. The results were reprinted by Blevins [14] and are neat and amazing; that is, the simplicity and order of their structure is so marvelous that these represent an example of the close relationship between mathematics, physics and beauty. Incidentally, it seems that Sharma [15] was not aware of this preceding work while developing a paper on the same subject. By contrast, in another work Leung [16] extended the original by Felgar; indeed, some erroneous results were corrected. However, integrals in their simplest form were not attained as in Felgar's work; actually, the objective of Leung's investigation was to generalize Felgar's results and to develop a computational algorithm for obtaining definite integrals. Additional integrals of that type in their simplest form were presented by Morales and Ramírez [17]; in addition, some integrals given by Blevins [14] that can be written in an even simpler form were also presented therein.

We present novel integrals in their simplest form involving beam-column eigenfunctions and derivatives. All the integrals arise in computational stability analysis of frames. It is emphasized that simpler results improve the accuracy of Rayleigh–Ritz-based approximate methods for buckling analysis of distributed structures.

2. Definitions and results

The beam-column eigenfunctions satisfy the ordinary equation

(2.1) $$Y'''' + k^2 Y'' = 0,$$

wherein the independent variable is the spatial variable x and k is given by

(2.2) $$k^2 = \frac{P}{EI},$$

where P is the axial compressive load and EI is the flexural rigidity. The characteristic functions for clamped-clamped, clamped-pinned, pinned-pinned, clamped-free and pinned-clamped beams are, respectively, given by

$$
\begin{aligned}
Y_{cc} &= \cos k_1 x - 1 + \sigma_1 (k_1 x - \sin k_1 x), \\
Y_{cp} &= \sin k_2 x + k_2 (L(1 - \cos k_2 x) - x), \\
Y_{pp} &= \sin k_3 x, \\
Y_{cf} &= \cos k_4 x - 1, \\
Y_{pc} &= \sin k_2 x - \sigma_2 k_2 x,
\end{aligned}
$$

(2.3)

where the characteristic k is defined by

$$
\begin{aligned}
2(1 - \cos k_1 L) &= k_1 L \sin k_1 L, \\
\tan k_2 L &= k_2 L, \\
\sin k_3 L &= 0, \\
\cos k_4 L &= 0,
\end{aligned}
$$

(2.4)

and the characteristic sigmas by

(2.5) $$\sigma_1 = \frac{\cos k_1 L - 1}{\sin k_1 L - k_1 L}, \qquad \sigma_2 = \cos k_2 L,$$

where L is the beam length. Finally, in the table of integrals that follows, the indices r and s indicate the mode and the primes indicate derivatives.

The integrals:

1. $\displaystyle\int_0^L Y_{pp_r}^2 = \frac{1}{2} L.$

2. $\displaystyle\int_0^L Y_{cp_r}'' Y_{cp_r}'' = \frac{1}{2} k_{2r}^6 L^3.$

3. $\displaystyle\int_0^L Y''_{cp_r} Y''_{cc_s} = \frac{k_{1s}k_{2r}^3}{(k_{2r}^2 - k_{1s}^2)}(k_{1s}^2 L\sigma_{1s} - k_{1s})(\sigma_{2r}(k_{2r}^2 L^2 + 1) - 1).$

4. $\displaystyle\int_0^L Y''_{pp_r} Y''_{pp_r} = \frac{1}{2}k_{3r}^4 L.$

5. $\displaystyle\int_0^L Y''_{pp_r} Y''_{cp_s} = \frac{k_{2s}^3 k_{3r}^3 L}{k_{2s}^2 - k_{3r}^2}.$

6. $\displaystyle\int_0^L Y''_{cc_r} Y''_{cc_r} = \frac{1}{2}k_{1r}^4 L.$

7. $\displaystyle\int_0^L Y''_{cc_r} Y''_{pp_s} = \frac{k_{1r}k_{3s}^3}{k_{3s}^2 - k_{1r}^2}((-1)^s(k_{1r}^2 L\sigma_{1r} - k_{1r}) + k_{1r}).$

8. $\displaystyle\int_0^L Y''_{cc_r} Y''_{cf_s} = \frac{k_{1r}k_{4s}^2}{k_{4s}^2 - k_{1r}^2}(k_{1r}^2\sigma_{1r} - k_{4s}(k_{1r}^2 L\sigma_{1r} - k_{1r})(-1)^{s+1}).$

9. $\displaystyle\int_0^L Y''_{cf_r} Y''_{cf_r} = \frac{1}{2}k_{4r}^4 L.$

10. $\displaystyle\int_0^L Y''_{pc_r} Y''_{pc_r} = \frac{1}{2}k_{2r}^6 L^3 \sigma_{2r}^2.$

11. $\displaystyle\int_0^L Y''_{cp_r} Y''_{pc_r} = 0.$

12. $\displaystyle\int_0^L Y''_{cp_r} Y''_{pc_s} = \frac{k_{2r}^3 k_{2s}^3 L}{k_{2r}^2 - k_{2s}^2}(1 - \sigma_{2r}\sigma_{2s}(k_{2r}^2 L^2 + 1)) \quad \text{for } r \neq s.$

13. $\displaystyle\int_0^L Y'_{cp_r} Y'_{cp_r} = \frac{1}{2}k_{2r}^4 L^3.$

14. $\displaystyle\int_0^L Y'_{cp_r} Y'_{cc_s} = \frac{k_{1s}k_{2r}}{(k_{2r}^2 - k_{1s}^2)}(k_{1s}^2 L\sigma_{1s} - k_{1s})(\sigma_{2r}(k_{2r}^2 L^2 + 1) - 1).$

15. $\displaystyle\int_0^L Y'_{pp_r} Y'_{pp_r} = \frac{1}{2}k_{3r}^2 L.$

16. $\int_0^L Y'_{pp_r} Y'_{cp_s} = \dfrac{k_{2s}^3 k_{3r} L}{k_{2s}^2 - k_{3r}^2}.$

17. $\int_0^L Y'_{cc_r} Y'_{cc_r} = \dfrac{1}{2} k_{1r}^2 L.$

18. $\int_0^L Y'_{cc_r} Y'_{pp_s} = \dfrac{k_{1r} k_{3s}}{k_{3s}^2 - k_{1r}^2} ((-1)^s (k_{1r}^2 L \sigma_{1r} - k_{1r}) + k_{1r}).$

19. $\int_0^L Y'_{cc_r} Y'_{cf_s} = \dfrac{k_{1r}}{k_{4s}^2 - k_{1r}^2} (k_{1r}^2 \sigma_{1r} - k_{4s} (k_{1r}^2 L \sigma_{1r} - k_{1r})(-1)^{s+1}).$

20. $\int_0^L Y'_{cf_r} Y'_{cf_r} = \dfrac{1}{2} k_{4r}^2 L.$

21. $\int_0^L Y'_{pc_r} Y'_{pc_r} = \dfrac{1}{2} k_{2r}^2 L \left(2 - \sigma_{2r}^2 \left(k_{2r}^2 L^2 + 2\right)\right).$

22. $\int_0^L Y'_{cp_r} Y'_{pc_r} = k_{2r}^2 L \left(1 - \sigma_{2r}\right).$

23. $\int_0^L Y'_{cp_r} Y'_{pc_s} = \dfrac{k_{2r} k_{2s} L}{k_{2r}^2 - k_{2s}^2} (k_{2r}^2 - (k_{2r}^2 - k_{2s}^2 + k_{2s}^2 (k_{2r}^2 L^2 + 1)\sigma_{2r})\sigma_{2s})$ for $r \neq s$.

3. Remarks on these and previous results

Apart from assuring neatness in mechanics, the practical implication of this work and the precedent ones [14, 17] is that simple and computer-friendly expressions for the characteristic integrals have been attained. The results are important because complex approximate methods such as the ones based on the Rayleigh–Ritz theory, including the finite element method, ask for easy system matrices construction and for curtailing the number of both computer operations and subordinate numerical methods in order to protect accuracy. In other words, the interest is in the actual convergence of the approximate technique to be controlled only by its inherent numerical characteristics and not by round-off errors associated with operations involving trigonometric and hyperbolic functions, for example, or secondary approximate methods such as numerical integration. Other contributions [16] do not ensure that the Rayleigh–Ritz procedures based on the algorithms presented therein are numerically optimal

in the sense explained before. Finally, the integrals were of course obtained by hand; it must be said that approximately half of the table could be obtained through an advanced computer algebra software (like Mathematica®) but it would require more steps and commands than the basic ones, more importantly, it would require the assumption of equations that are presented in this Note or in specialized literature on structural stability; that is, some of the results presented could be obtained by symbolic manipulation software but it demands more than beginner's knowledge of the software and reading or studying of beam-column eigenfuctions and their mathematics. The other half simply cannot be obtained, in the simplest or most elegant form, through purely specialized software.

4. Conclusions

Simplest-expression integrals that contain eigenfunctions of Euler–Bernoulli beam-column buckling boundary-value problems have been obtained; the expressions are written in terms of beam length and a few characteristic constants (L, k_i and σ_i). This type of integrals appears in optimal structural applications of Rayleigh–Ritz methodologies; those prove to be powerful for simplifying these techniques and improving their convergence characteristics; initial results on this matter are being obtained at this point.

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