

## The fractional effects of a two-temperature generalized thermoelastic semi-infinite solid induced by pulsed laser heating

A. M. ZENKOUR<sup>1), 2)</sup>, A. E. ABOUELREGAL<sup>3), 4)</sup>

<sup>1)</sup> *Department of Mathematics  
Faculty of Science, King Abdulaziz University  
P.O. Box 80203, Jeddah 21589, Saudi Arabia  
e-mail: zenkour@kau.edu.sa*

<sup>2)</sup> *Department of Mathematics  
Faculty of Science, Kafrelsheikh University  
Kafr El-Sheikh 33516, Egypt  
e-mail: zenkour@kaf-sci.edu.eg*

<sup>3)</sup> *Department of Mathematics  
Faculty of Science, Mansoura University  
Mansoura 35516, Egypt  
e-mail: ahabogal@gmail.com*

<sup>4)</sup> *Department of Mathematics  
College of Science and Arts  
University of Aljouf, El-Qurayat, Saudi Arabia*

IN THIS PAPER, A THEORY OF TWO-TEMPERATURE GENERALIZED thermoelasticity is constructed in the context of a new consideration of heat conduction with fractional orders. The obtained general solution is applied to a specific problem of a medium, semi-infinite solid considered to be made of a homogeneous thermoelastic material. The bounding plane surface of the medium is being subjected to a non-Gaussian laser pulse. The medium is assumed initially quiescent and Laplace transforms techniques will be used to obtain the general solution for any set of boundary conditions. The inverse of the Laplace transforms are computed numerically using a method based on Fourier's expansion techniques. The theories of coupled thermoelasticity and of generalized thermoelasticity with one relaxation time follow as limit cases. Some comparisons have been shown in figures to estimate the effects of the fractional order, temperature discrepancy, laser-pulse and the laser intensity parameters on all the studied fields.

**Key words:** generalized thermoelasticity, two temperatures, fractional order, non-Gaussian laser pulse, laser intensity.

Copyright © 2015 by IPPT PAN

## 1. Introduction

THE EFFECT OF TEMPERATURE ON SOIL BEHAVIOUR is a crucial problem in some important cases, i.e., the storage of hot fluids, the disposal of radioactive waste and that of explosive shock. The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; secondly, the heat equation is of a parabolic type, predicting infinite speeds of propagation for thermal signals. Thus, the classical uncoupled theory of thermoelasticity contradicts the physical facts.

BIOT [1] has introduced a theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, this theory shares the second shortcoming since the heat equation for the coupled theory is based on Fourier's law of heat conduction and it is also parabolic. Henceforth, two styles of generalized theories of thermoelasticity presented by LORD and SHULMANN [2], and GREEN and LINSLEY [3], which consider the finite speed of the thermal signal, have been the center of interest of active research during the last three decades. The third generalization to the coupled theory is known as the dual-phase-lag thermoelasticity, proposed by TZOU [4], in which Fourier's law is replaced by an approximation to a modification of Fourier's law with two different translations for the heat flux and the temperature gradient. These theories remove the paradox of the infinite speed of heat propagation that is inherent in the conventional coupled dynamical theory of thermoelasticity, which was introduced by BIOT [1]. In the generalized theories, a modified heat conduction law, which includes both the heat flux and its time derivative, replaced the conventional Fourier's law. GREEN and NAGHDI [5]–[7] have introduced a new theory of thermoelasticity and divided their theory into three parts, referred as types I, II and III.

CHEN and his colleagues [8]–[10] have formulated a theory of heat conduction in deformable bodies, which depends on two distinct temperatures, the conductive temperature and thermodynamic temperature. For time-independent situations, the difference between these two temperatures is proportional to the heat supply, and in the absence of any heat supply, the two temperatures are identical. For time-dependent problems, however, and for wave propagation problems in particular, the two temperatures are in general different regardless of the presence of a heat supply. The two temperatures and the strain are found to appear in the form of a traveling wave plus a response, which occurs instantaneously throughout the body [11]. WARREN and CHEN [12] have investigated the wave propagation in the two-temperature thermoelasticity theory (2TT), but YOUSSEF [13] has extended this theory in the context of the generalized

theory of thermoeasticity. He has improved the previous theories of the generalized thermoelasticity by introducing a new theory with two temperatures called two-temperature generalized thermoelasticity. Recently, the first author and his colleagues have presented additional investigations concerning the two-temperature model [14]–[17].

Excitation of thermoelastic waves by a pulsed laser in solid is of great interest due to extensive applications of pulsed laser technologies in material processing and nondestructive detecting and characterization. When a solid is illuminated with a laser pulse, absorption of the laser pulse results in a localized temperature increase, which in turn causes thermal expansion and generates a thermoelastic wave in the solid. In ultrashort-pulsed laser heating, two effects become important. Due to the extremely short heating time, modifications to the Fourier heat conduction theory are necessary in order to predict the temperature field correctly. One modification is to consider the non-Fourier effect, which accounts for a thermal relaxation time in the energy carrier's collision process. The other modification is to employ the two-temperature (or two-step heat transfer) model [18] to account for the fact that the photon energy is first absorbed by the electron gas in a metal and is then transferred to the lattice. Using this model, the temperature of the electron gas was found to be much higher than that of the lattice during the initial period of ultrafast-laser heating [19, 20].

Fractional calculus has been used successfully to modify many existing models of physical process. In the formulation of tautochrone problem, Abel applied fractional calculus to solve the integral equation and that was the first application of fractional derivatives. In the second half of the nineteenth century, CAPUTO [21] and CAPUTO and MAINARDI [22] have found an agreement between the experimental results with theoretical ones when using fractional derivatives for the description of viscoelastic materials. POVSTENKO [23] has proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative of order  $\alpha$ . In [24] POVSTENKO has investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. JIANG and XU [25] have obtained a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate and in other orthogonal coordinate system. ABOUELREGAL and ZENKOUR [26] have presented the effect of fractional thermoelasticity on a two-dimensional problem of a mode I crack in a rotating fibre-reinforced thermoelastic medium. ABBAS and ZENKOUR [27] have presented the semi-analytical and numerical solutions of fractional order in generalized thermoelastic in a semi-infinite medium.

The current manuscript is an attempt to study the induced temperature and stress fields in an elastic half-space. The governing equations are written in the context of two-temperature generalized and fractional order thermoelasticity the-

ories. The half-space is considered to be made of an isotropic homogeneous thermoelastic material. The bounding plane surface is heated by a non-Gaussian laser beam. An exact solution of the problem is first obtained in Laplace transform space. The inversion of the Laplace transform will be computed numerically by using a method based on Fourier's expansion technique. The numerical estimates of the conductive temperature, the thermodynamic temperature, the stress and the strain distributions are obtained. The derived expressions are computed numerically for copper and the results are presented in graphical form. The effects of the fractional parameter, the two-temperature parameter, and the laser-pulse and the laser intensity parameters are estimated. Some special cases are also considered.

## 2. Governing equation of two-temperature with fractional order

During recent years, several interesting models have been developed by using fractional calculus to study the physical processes particularly in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity, dielectrics and semiconductors through polymers to fractals, glasses, porous, and random media, porous glasses, polymer chains and biological systems. It has been found that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems, which is more economical and useful than the classical approach. There exists many material and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading etc., where the classical thermoelasticity based on Fourier's type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time fractional (non-integer order) derivatives.

The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type integral [23, 24, 28, 29]:

$$(2.1) \quad I^\alpha f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad \alpha > 0,$$

where  $f(t)$  is a Lebesgue integrable function,  $\Gamma(\alpha)$  is the gamma function and  $t$  is the time. In the case that  $f(t)$  is absolutely continuous, then

$$(2.2) \quad \lim_{\alpha \rightarrow 1} \frac{d^\alpha}{dt^\alpha} f(t) = f'(t).$$

The classical thermoelasticity is based on the principles of the classical theory of heat conductivity, specifically on the classical Fourier law, which relates the

heat flux vector  $\mathbf{q}$  to the temperature gradient as follows:

$$(2.3) \quad \mathbf{q} = -K\nabla T,$$

where  $K$  is the thermal conductivity of a solid, which together with the energy equation yields the heat conduction equation or the parabolic heat conduction equation and is diffusive with the notion of infinite speed of propagation of thermal disturbances,

$$(2.4) \quad \rho C_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) = -\nabla \cdot \mathbf{q} + Q,$$

where  $\rho$  is the density,  $C_E$  is the specific heat,  $T_0$  is the reference temperature,  $\mathbf{u}$  is the displacement vector,  $\gamma = (3\lambda + 2\mu)\alpha_t$ ,  $\lambda$  and  $\mu$  being Lamé's constants,  $\alpha_t$  being the coefficient of linear thermal expansion and  $Q$  is the intensity of heat source.

EZZAT [30] has constructed a new model of the magneto-thermoelasticity theory in the context of a new consideration of heat conduction equation by using the Taylor series expansion of time fractional order, developed by JUMARIE [31] as

$$(2.5) \quad \mathbf{q} + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -K\nabla T.$$

Now, in isotropic media, we assume a new generalized heat conduction equation of the form

$$(2.6) \quad \mathbf{q} + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -K\nabla \varphi,$$

where  $\varphi$  is the conductive temperature that satisfies the relation

$$(2.7) \quad \varphi - T = b\varphi_{,ii},$$

where  $b > 0$  is the two-temperature parameter. Now, taking divergence of both sides of Eq. (2.6), we get

$$(2.8) \quad (\nabla \cdot \mathbf{q}) + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} (\nabla \cdot \mathbf{q}) = -K\nabla^2 \varphi.$$

Using Eq. (2.4), we obtain

$$(2.9) \quad K\nabla^2 \varphi = \left( \delta + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left( \rho C_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) - Q \right),$$

where  $\delta$  is an arbitrary parameter. The above equation may be considered as the fractional ordered generalized heat conduction equation in isotropic, thermoelastic solids in two temperatures. The key element that sets the two-temperature

thermoelasticity theory apart from the classical theory is the material parameter  $b$ . Specifically, in the limit as  $b \rightarrow 0$ ,  $\varphi \rightarrow \theta$ , and  $\alpha \rightarrow 1$  the classical theory (one-temperature generalized thermoelasticity theory 1TT) is recovered.

The equations of motion without body forces take the form

$$(2.10) \quad (\lambda + \mu)u_{i,ij} + \mu u_{i,jj} - \gamma \theta_{,i} = \rho \ddot{u}_i,$$

where  $\theta = T - T_0$  denotes the thermodynamical temperature. The constitutive equation takes the form

$$(2.11) \quad \sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk} \delta_{ij} - \gamma(T - T_0)\delta_{ij},$$

where  $\sigma_{ij}$  is the stress tensor and  $\delta_{ij}$  is the Kronecker delta function.

The following special cases can be obtained from the system of Eqs. (2.7) and (2.9)–(2.11):

- The equations of a coupled theory of two-temperature thermoelasticity (CTE):  $\tau_0 = 0$  and  $\delta = 1$ .
- The equations of a generalized theory of two-temperature thermoelasticity (LS):  $\alpha \rightarrow 1$  and  $\delta = 1$ .
- The equations of a generalized two-temperature thermoelasticity without energy dissipation (GN):  $\tau_0 = 1$ ,  $\alpha \rightarrow 1$  and  $\delta = 0$ .
- The equations of one-temperature generalized thermoelasticity theories with fractional order heat conduction (CTE, LS and GN):  $b \rightarrow 0$  and  $\varphi \rightarrow \theta$ .
- The equations of one-temperature generalized thermoelasticity theories without fractional heat conduction (CTE, LS and GN):  $b \rightarrow 0$ ,  $\varphi \rightarrow \theta$  and  $\alpha \rightarrow 1$ .

### 3. Statement of the problem

Now, let us consider a homogeneous thermoelastic conducting isotropic solid occupying a half-space  $x \geq 0$ . This half-space is irradiating uniformly the bounding plane ( $x = 0$ ) by a laser pulse with non-Gaussian temporal profile. The system is initially quiescent where all the state functions are depending only on the variable  $x$  and the time  $t$ .

The displacement components for one-dimensional medium have the forms  $u_x = u(x, t)$  and  $u_y = u_z = 0$ . The relation between the strain and displacement can be expressed as  $e = e_{xx} = \partial u / \partial x$ .

The fractional heat conduction equation [10] is given by

$$(3.1) \quad K \frac{\partial^2 \varphi}{\partial x^2} = \left( \delta + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left( \rho C_E \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t} - Q \right).$$

The constitutive equation will be

$$(3.2) \quad \sigma_{xx} = \sigma = (\lambda + 2\mu)e - \gamma\theta.$$

The equation of motion takes the form

$$(3.3) \quad (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial \theta}{\partial x} = \rho\ddot{u}_i,$$

or may be written in the form

$$(3.4) \quad \frac{\partial^2 \sigma}{\partial x^2} = \rho\frac{\partial^2 e}{\partial t^2}.$$

The relation between the heat conduction and the thermodynamic heat takes the form

$$(3.5) \quad \varphi - \theta = b\frac{\partial^2 \varphi}{\partial x^2}.$$

Next, we introduce the following non-dimensional parameters:

$$(3.6) \quad \begin{aligned} x' &= c_1 \eta x, & u' &= c_1 \eta u, & \tau'_0 &= c_1^2 \eta \tau_0, & t' &= c_1^2 \eta t, \\ \theta' &= \frac{\gamma \theta}{\rho c_1^2}, & \varphi' &= \frac{\gamma \varphi}{\rho c_1^2}, & \sigma' &= \frac{\sigma}{\rho c_1^2}, & Q' &= \frac{Q}{K c_1^2 \eta^2 T_0}, \end{aligned}$$

where  $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$  is the longitudinal wave speed and  $\eta = \rho C_E/K$  is the thermal viscosity. Then, Eqs. (3.1)–(3.5) can be transformed into the dimensionless forms:

$$(3.7) \quad \frac{\partial^2 \varphi}{\partial x^2} = \left( \delta + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left( \frac{\partial \theta}{\partial t} + \varepsilon \frac{\partial e}{\partial t} - Q \right),$$

$$(3.8) \quad \frac{\partial^2 e}{\partial x^2} - \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 e}{\partial t^2},$$

$$(3.9) \quad \varphi - \theta = \beta \frac{\partial^2 \varphi}{\partial x^2},$$

where  $\varepsilon = \gamma^2 T_0 / \rho^2 C_E c_1^2$  and  $\beta = b c_1^2 \eta^2$ . Now, let the medium is heated uniformly by a laser pulse with non-Gaussian form temporal profile [18] as

$$(3.10) \quad I(t) = \frac{L_0 t}{t_p^2} e^{-t/t_p},$$

where  $t_p$  is a characteristic time (measured by picoseconds) of the laser-pulse (the time duration of a laser pulse),  $L_0$  is the laser intensity which is defined

as the total energy carried by a laser pulse per unit area of the laser beam. The conduction heat transfer in the medium can be modeled as one-dimensional problem with an energy source  $Q(x, t)$  near the surface, i.e.,

$$(3.11) \quad Q(x, t) = \frac{1-R}{\delta_1} e^{(x-h/2)/\delta_1} I(t) = \frac{R_a L_0}{\delta_1 t_p^2} t e^{(x-h/2)/\delta_1 - t/t_p},$$

where  $\delta_1$  is the absorption depth of heating energy and  $R_a$  is the surface reflectivity [18]. Note that the laser pulse may lie on the surface of the medium ( $x = 0$ ). In this case, the energy source takes the form

$$(3.12) \quad Q(t) = \frac{R_a L_0}{\delta_1 t_p^2} t e^{h/(2\delta_1) - t/t_p}.$$

#### 4. Initial and boundary conditions

The problem is solved under proper initial and boundary conditions. The metal film is initially unstrained, unstressed and at ambient temperature  $T_0$  throughout. That is, the initial conditions are:

$$(4.1) \quad \begin{aligned} \theta(x, t) = \varphi(x, t) = u(x, t) &= 0 & \text{at } t = 0, \\ \frac{\partial \theta(x, t)}{\partial t} = \frac{\partial \varphi(x, t)}{\partial t} = \frac{\partial u(x, t)}{\partial t} &= 0 & \text{at } t = 0. \end{aligned}$$

The thermal and mechanical boundary conditions on the bounding plane  $x = 0$  of the assumed half-space are given as follows:

- Thermal boundary condition:  $\varphi(0, t) = 0$ .
- Mechanical boundary condition:  $\sigma(0, t) = 0$ .

#### 5. Exact solution in the Laplace transform domain

Applying Laplace transform with respect to variable  $t$  for Eqs. (3.7)–(3.9) and (2.11) defined by the formula

$$(5.1) \quad \bar{f}(s) = \int_0^\infty f(t) e^{-st} dt.$$

Hence, one can get the system of differential equations in the transformed domain as follows:

$$(5.2) \quad \frac{d^2 \bar{\varphi}}{dx^2} = \alpha_1 (\bar{\theta} + \varepsilon \bar{e}) - \bar{G}(s),$$

$$(5.3) \quad \left( \frac{d^2}{dx^2} - s^2 \right) \bar{e} = \frac{d^2 \bar{\theta}}{dx^2},$$



$$(5.4) \quad \bar{\theta} = \bar{\varphi} - \beta \frac{d^2 \bar{\varphi}}{dx^2},$$

$$(5.5) \quad \bar{\sigma} = \bar{e} - \bar{\theta},$$

where

$$(5.6) \quad \bar{G}(s) = \frac{\gamma R_a L_0 \alpha_1}{K c_1 \delta_1 (1 + s t_p^2)^2} e^{-h/(2\delta_1)}, \quad \alpha_1 = \delta + \frac{\tau_0^\alpha s^\alpha}{\alpha!}.$$

Eliminating  $\bar{\theta}$  and  $\bar{e}$  from these equations, one obtains

$$(5.7) \quad \left( \frac{d^4}{dx^4} - A \frac{d^2}{dx^2} + B \right) \bar{\varphi} = -F(s),$$

where

$$(5.8) \quad A = \frac{s^2(1 + \alpha_1 \beta) + \alpha_1(1 + \varepsilon)}{1 + \alpha_1 \beta(1 + \varepsilon)}, \quad B = \frac{s^2 \alpha_1}{1 + \alpha_1 \beta(1 + \varepsilon)},$$

$$F(s) = \frac{s^2 \bar{G}(s)}{1 + \alpha_1 \beta(1 + \varepsilon)}.$$

The solution of Eq. (5.7) takes the following form:

$$(5.9) \quad \bar{\varphi} = -\frac{F(s)}{B} + A_1 e^{-m_1 x} + A_2 e^{-m_2 x},$$

where  $A_1$  and  $A_2$  are parameters of  $s$ . In a similar manner, one obtains

$$(5.10) \quad \left( \frac{d^4}{dx^4} - A \frac{d^2}{dx^2} + B \right) \bar{e} = 0,$$

thus,

$$(5.11) \quad \bar{e} = B_1 e^{-m_1 x} + B_2 e^{-m_2 x},$$

where  $B_1$  and  $B_2$  are additional parameters of  $s$ . Substituting Eqs. (5.9) and (5.11) into Eqs. (5.4) and (5.5), one obtains

$$(5.12) \quad \bar{\theta} = -\frac{F(s)}{B} + (1 - \beta m_1^2) A_1 e^{-m_1 x} + (1 - \beta m_2^2) A_2 e^{-m_2 x},$$

$$(5.13) \quad \bar{\sigma} = [B_1 - (1 - \beta m_1^2) A_1] e^{-m_1 x} + [B_2 - (1 - \beta m_2^2) A_2] e^{-m_2 x} + \frac{F(s)}{B}.$$

Substituting Eqs. (5.11) and (5.12) into Eq. (5.3), one can get

$$(5.14) \quad B_i = \frac{m_i^2(1 - \beta m_i^2)}{m_i^2 - s^2} A_i = \Omega_i A_i, \quad i = 1, 2.$$

In addition, the thermal and mechanical boundary conditions in the Laplace domain  $\bar{\varphi}(0, s) = 0$  and  $\bar{\sigma}(0, s) = 0$  with the aid of Eqs. (5.9) and (5.13), give

$$(5.15) \quad A_1 = -\frac{F(s)(\Omega_2 + \beta m_2^2)}{B[\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)]}, \quad A_2 = \frac{F(s)(\Omega_1 + \beta m_1^2)}{B[\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)]}.$$

So, one can write the solutions in their final forms as

$$(5.16) \quad \bar{u} = \frac{F(s)[m_2\Omega_1(\Omega_2 + \beta m_2^2)e^{-m_1x} - m_1\Omega_2(\Omega_1 + \beta m_1^2)e^{-m_2x}]}{Bm_1m_2[\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)]},$$

$$(5.17) \quad \bar{\theta} = -\frac{F(s)}{B} \times \left[ 1 + \frac{(1 - \beta m_1^2)(\Omega_2 + \beta m_2^2)e^{-m_1x} - (1 - \beta m_2^2)(\Omega_1 + \beta m_1^2)e^{-m_2x}}{\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)} \right],$$

$$(5.18) \quad \bar{\varphi} = -\frac{F(s)}{B} \left[ 1 + \frac{(\Omega_2 + \beta m_2^2)e^{-m_1x} - (\Omega_1 + \beta m_1^2)e^{-m_2x}}{\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)} \right],$$

$$(5.19) \quad \bar{\sigma} = \frac{F(s)}{B} \times \left[ 1 - \frac{(\Omega_1 - 1 + \beta m_1^2)(\Omega_2 + \beta m_2^2)e^{-m_1x} - (\Omega_2 - 1 + \beta m_2^2)(\Omega_1 + \beta m_1^2)e^{-m_2x}}{\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)} \right],$$

$$(5.20) \quad \bar{e} = -\frac{F(s)[\Omega_1(\Omega_2 + \beta m_2^2)e^{-m_1x} - \Omega_2(\Omega_1 + \beta m_1^2)e^{-m_2x}]}{B[\Omega_1 - \Omega_2 + \beta(m_1^2 - m_2^2)]}.$$

This completes the solution in the Laplace transform domain.

## 6. Numerical inversion of the Laplace transform

Laplace transform method is efficient for solving partial differential equations. The present research utilizes Laplace transform to suppress the dependence on time concerning the transient response of beam under harmonic heat with constant angular frequency. In the transformation domain, the boundary value problems are solved. To invert the solutions to the physical domain, the inversion of Laplace transform must be made. However, the expressions of the solutions are usually complicated and cannot be inverted analytically. As an alternative, the numerical inversion is applied. Presently, over 20 methods have been developed for the inversion.

In order to determine the conductive and thermal temperature as well as displacement and stress distributions in the time domain, we adopt a numerical inversion method based on a Fourier series expansion [32]. In this method, the

inverse  $f(t)$  of the Laplace transform  $\bar{f}(s)$  is approximated by the relation

$$(6.1) \quad f(t) = \frac{e^{\zeta t}}{t_1} \left( \frac{1}{2} \bar{f}(\zeta) + \operatorname{Re} \left\{ \sum_{k=0}^N \bar{f} \left( \zeta + \frac{ik\pi t/t_1}{t_1} \right) e^{ik\pi t/t_1} \right\} \right), \quad 0 \leq t \leq t_1,$$

where  $\operatorname{Re}$  is the real part and  $i$  is imaginary number unit and  $N$  is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series. It must be chosen such that

$$(6.2) \quad e^{\zeta t} e^{iN\pi t/t_1} \operatorname{Re} \left\{ \bar{f} \left( \zeta + \frac{iN\pi}{t_1} \right) \right\} \leq \varepsilon_1,$$

where  $\varepsilon_1$  is a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter  $\zeta$  is a positive free parameter that must be greater than the real parts of all singularities of  $\bar{f}(s)$ . The optimal choice of  $\zeta$  was obtained according to the criteria described in [32].

## 7. Numerical results and discussion

To study the effect of the two-temperature parameter, laser-pulse and the laser intensity coefficients and fractional parameters on wave propagation, the following physical constants for copper material are used:

$$K = 368 \text{ N/Ks}, \quad \alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, \quad C_E = 383.1 \text{ m}^2/\text{K}, \quad \rho = 8954 \text{ kg/m}^3, \\ \lambda = 7.76 \times 10^{10} \text{ N/m}^2, \quad \mu = 3.86 \times 10^{10} \text{ N/m}^2, \quad T_0 = 293 \text{ K}.$$

The computations were carried out for wide range of  $x$  ( $0 \leq x \leq 1$ ) at small value of time  $t = 0.15$  s. The physical quantities are plotted in Figs. 1–15. For all numerical calculations one takes  $\delta_1 = 0.01$ ,  $\tau_0 = 0.02$ ,  $R_a = 0.5$  and  $h = 0.1$ . The field quantities such as the conductive temperature, the dynamical temperature, the stress, the strain, and the displacement distributions depend not only on the time  $t$  and space coordinate  $x$ , but also on the two-temperature parameter  $\beta$ , the time of the laser-pulse  $t_p$ , the laser intensity  $L_0$ , and the fractional order parameter  $\alpha$ . The laser intensity  $L_0$  is assumed to be of the form  $L_0 = c \times 10^{11} \text{ J/m}^2$  where  $c$  is the laser intensity parameter. Numerical calculation is carried out for three cases.

In the first case, Figs. 1–5 plot the displacement  $u$ , the thermo-dynamical temperature  $\theta$ , the conductive temperature  $\varphi$ , the stress  $\sigma$  and the strain  $e$  distributions with different values of the two-temperature parameter  $\beta$  to stand on the effect of this parameter on all of the studied fields. The value of  $\beta = 0.0$  indicates the old situation (one-temperature theory) while  $\beta = 0.2$  or  $0.4$  indicate the two-temperature theory. In this case one takes  $\alpha = 0.5$ ,  $c = 1$ , and  $t_p = 2$ .

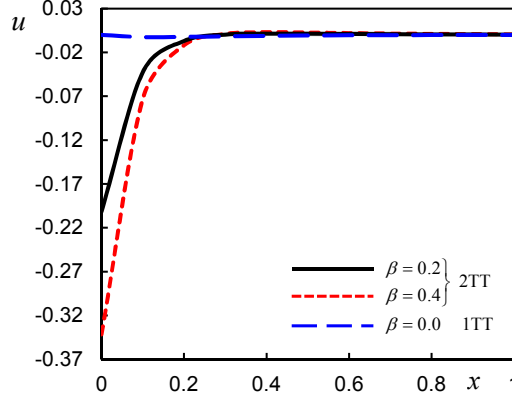


FIG. 1. Dependence of displacement  $u$  on the two temperature parameter  $\beta$ .

From Fig. 1, it can be observed that the medium adjacent to the half-space surface  $x = 0$  undergoes expansion deformation because of thermal shock while the others compressive deformation. The deformation is a dynamic process. With the passage of time, the expansion region moves inside gradually and becomes larger and larger. Thus, the displacement distribution becomes larger and larger. At a given instant, the non-zero region of displacement is finite, which is due to the wave effect of heat. This indicates that heat transfers deep into the medium with a finite velocity with the time passing. We can see also that the displacement decreases when the value of two-temperature parameter  $\beta$  increases.

Figure 2 indicates variation of temperature versus distance  $x$ . It can be found that the temperature has a non-zero value only in a bounded region of space at a given instant. Outside this region the value vanishes and this means that the region has not undergone thermal disturbance yet. At different instants, the

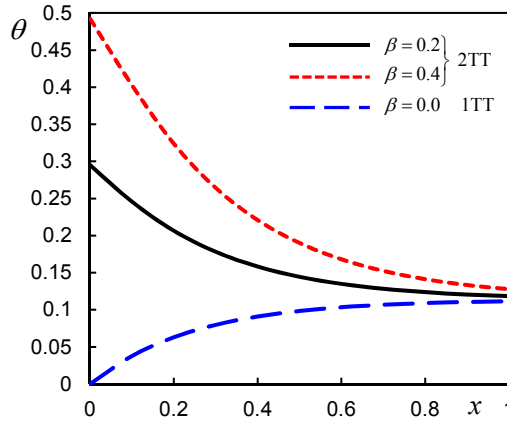


FIG. 2. Dependence of thermodynamical temperature  $\theta$  on the two temperature parameter  $\beta$ .

non-zero region moves forward correspondingly with the passage of time. This indicates that heat propagates as a wave with finite velocity in medium. This is completely different from the case for the classical theories of thermoelasticity where an infinite speed of propagation is inherent and hence all the considered functions have a non-zero (although may be very small) value for any point in the medium. Also these figures indicate that thermodynamic temperature increases when the value of two-temperature parameter  $\beta$  increases.

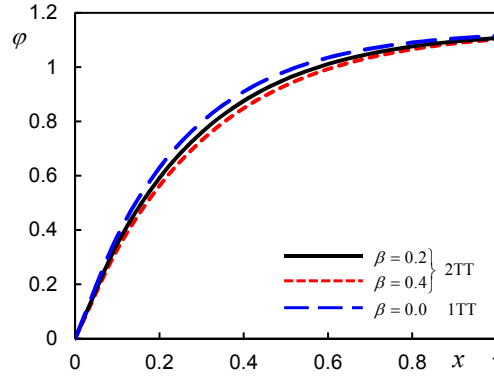


FIG. 3. Dependence of conductive temperature  $\varphi$  on the two temperature parameter  $\beta$ .

From Fig. 3, we notice that the conductive temperature  $\varphi$  gradually increases with distance  $x$  for fixed time. It also can be found that the conductive heat decreases when the value of  $\beta$  increases.

In Fig. 4, the thermal stress with distance  $x$  for various values of the two-temperature parameter  $\beta$  is shown. The medium close to the surface  $x = 0$  suffers from tensile stress. This is corresponding to the expansion deformation of the medium shown in Fig. 1. It can also be noticed that the tensile stress region becomes larger while the compressed becomes smaller with the time passing,

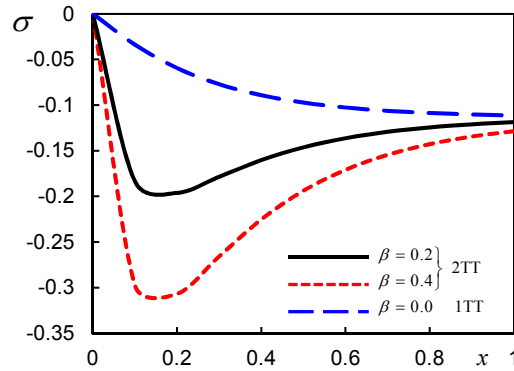


FIG. 4. Dependence of the thermal stress  $\sigma$  on the two temperature parameter  $\beta$ .

which is corresponding to the dynamic expansion effect described above. It can also be found in Fig. 4 that, at some instant, the non-zero region of stress is finite, which indirectly proves the wave effect of heat.

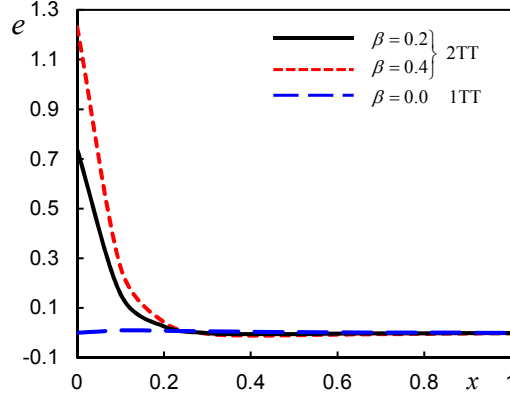
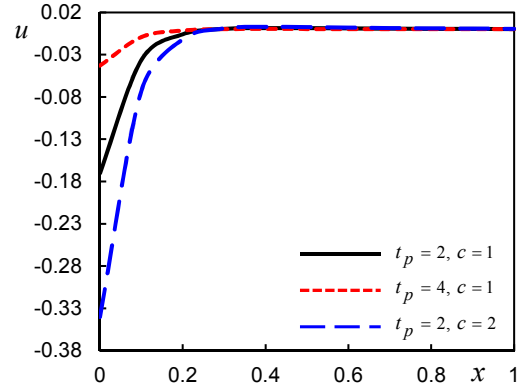
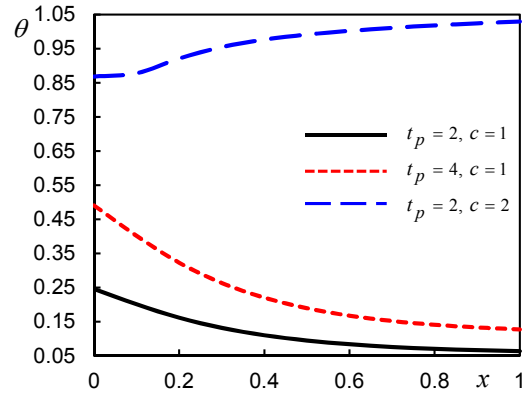
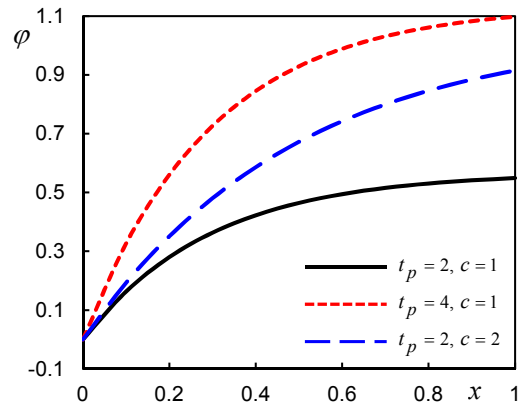


FIG. 5. Dependence of the strain  $e$  on the two temperature parameter  $\beta$ .

Figure 5 depicts the distributions of the strain  $e$  versus  $x$  at the boundary for different values of two-temperature parameter  $\beta$ . Also we can see that the strain decreases monotonically with  $x$ . Figure 5 shows that the values of strain  $e$  for  $\beta = 0.4$  are large compared with those for  $\beta = 0.2$ .

We can also mark the difference between the one-temperature generalized thermoelasticity of LORD and SHULMAN ( $\beta = 0.0$ ) and the two-temperature generalized thermoelasticity ( $\beta = 0.2$  or  $0.4$ ). The figures show that this parameter has significant effect on all the fields. The waves reach the steady state depending on the value of the temperature discrepancy  $\beta$ . Also these figures indicate that the two-temperature generalized theory of thermoelasticity describes the behavior of the particles of an elastic body more realistically than the one-temperature theory of generalized thermoelasticity.

In the second case, Figs. 6–10 plot the displacement  $u$ , the thermo-dynamical temperature  $\theta$ , the conductive temperature  $\varphi$ , the stress  $\sigma$  and the strain  $e$  distributions with the characteristic time of the laser-pulse  $t_p$  and the laser intensity parameter  $c$  to stand on the effect of these parameters on all studied fields. In this case the temperature discrepancy parameter  $\beta$  remains constant ( $\beta = 0.4$ ) and the fractional parameter  $\alpha = 0.5$ . It is found that the parameters  $t_p$  and  $c$  have significant effects on all fields. The figures show that the laser pulse makes the difference between the results in the context of the theory of two-temperature generalized thermoelasticity. The conductive heat and the thermodynamic heat increase when the value of  $t_p$  increases while the values of the displacement and stress decrease.

FIG. 6. Dependence of displacement  $u$  on the time of the laser-pulse and the laser intensity.FIG. 7. Dependence of thermodynamical temperature  $\theta$  on the time of the laser-pulse and the laser intensity.FIG. 8. Dependence of conductive temperature  $\varphi$  on the time of the laser-pulse and the laser intensity.

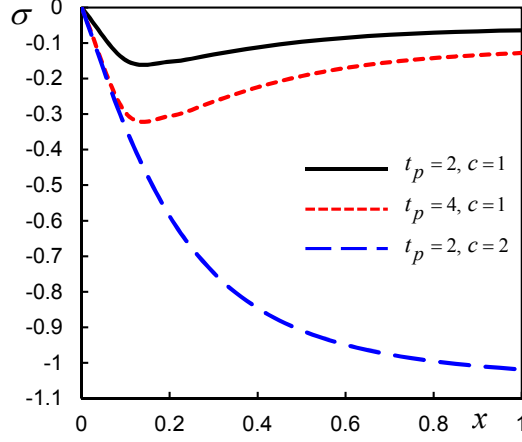


FIG. 9. Dependence of the thermal stress  $\sigma$  on the time of the laser-pulse and the laser intensity.

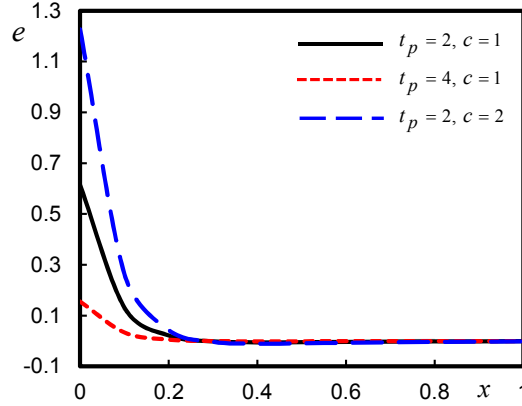


FIG. 10. Dependence of the strain  $e$  on the time of the laser-pulse and the laser intensity.

In the third case, Figs. 11–15 plot the displacement  $u$ , the thermo-dynamical temperature  $\theta$ , the conductive temperature  $\varphi$ , the stress  $\sigma$  and the strain  $e$  distributions at different fractional parameter  $\alpha$  to stand on the effect of this parameter on all the studied fields. It is found that this parameter has a significant effects on constancy of  $\beta = 0.4$ ,  $c = 1$  and  $t_p = 2$ . It is observed that the nature of variations of all field variables for fractional order parameter is significantly different. The different values of the parameter  $\alpha$  describe two types of conductivity (weak conductivity,  $0 < \alpha < 1$  and normal conductivity,  $\alpha = 1$ ), respectively. The difference is more prominent for higher values of fractional order  $\alpha$ . The following important facts are also observed:

- The displacement distribution shows a large change in comparison to fractional parameter  $\alpha$ .



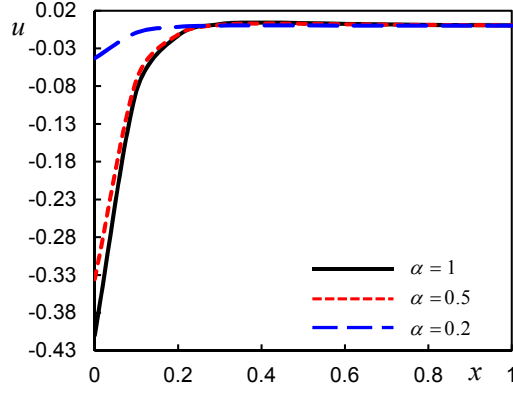


FIG. 11. Dependence of displacement  $u$  on the fractional order parameter  $\alpha$ .

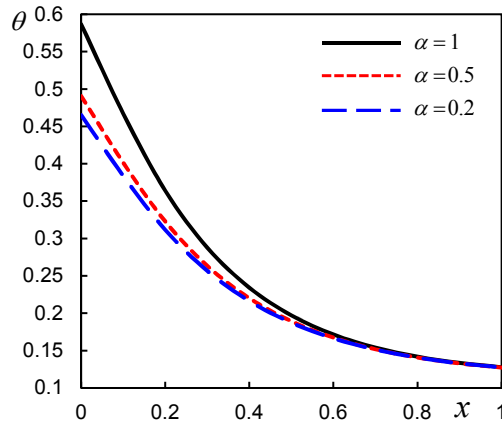


FIG. 12. Dependence of thermodynamical temperature  $\theta$  on the fractional order parameter  $\alpha$ .

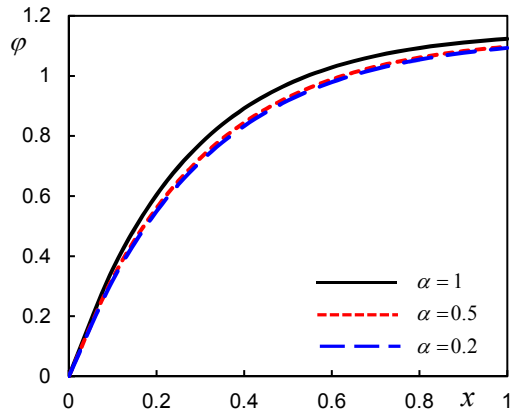


FIG. 13. Dependence of conductive temperature  $\varphi$  on the fractional order parameter  $\alpha$ .

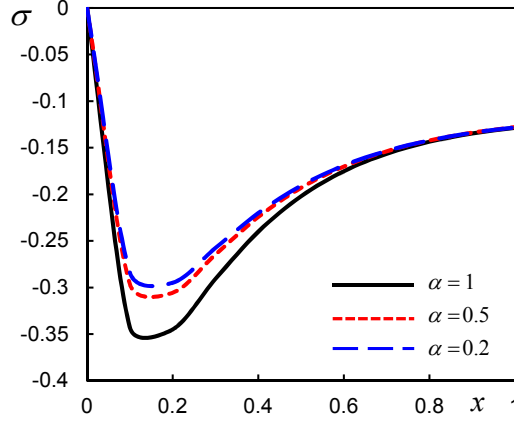


FIG. 14. Dependence of the thermal stress  $\sigma$  on the fractional order parameter  $\alpha$ .

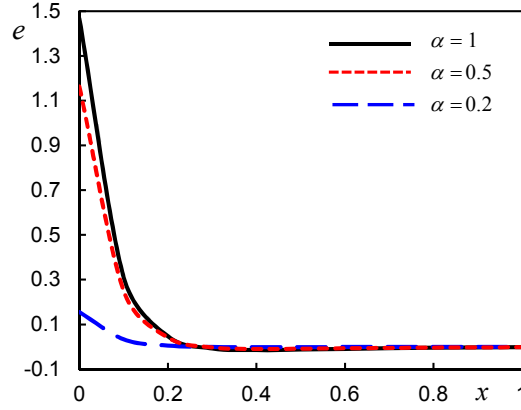


FIG. 15. Dependence of the strain  $e$  on the fractional order parameter  $\alpha$ .

- The temperature and stress distributions show large differences for different values of  $\alpha$ .
- Figure 13 indicates that the conductive temperature has a maximum value at boundary. It starts with zero value to satisfy the boundary conditions.
- In Fig. 14, the stress at  $x = 0$  is zero as shown, which agrees with the formulated boundary condition. This coincides with the mechanical boundary condition that the medium surface is traction-free.

## 8. Concluding remarks

In this work, a new mathematical model of generalized thermoelasticity theory has been studied in the context of a new consideration of heat conduction

with fractional order theory. This model is based on the heat conduction equation with the Caputo fractional derivative of order  $\alpha$ . The solution is obtained by applying the Laplace integral transform. The numerical results for temperature, displacement and stresses are computed and illustrated graphically. The results are graphically described for the medium of copper.

The analysis of the results can be summarized as follows:

- The dependence of the fractional parameter has a significant effect on the thermal and mechanical interactions, and plays a significant role in all the physical quantities.
- At any point, the distributions of temperature, conductive temperature and strain fields in the medium are increased with an increase in  $\alpha$  but the effect of fractional parameters is to decrease the values of the displacement and stress with a wide range ( $0 < \alpha \leq 1$ ).
- It is clear from Figs. 1–5 that different two-temperature parameters play a significant role in all the physical quantities.
- All the physical quantities satisfy the boundary conditions and initial conditions.
- It is also observed that the theories of coupled thermoelasticity and generalized thermoelasticity with one-relaxation time can be obtained as limited cases.
- The phenomenon of finite speeds of propagation is manifested in all these figures.
- As a final remark, the results presented in this paper should prove useful for researchers in material science, designers of new materials, low-temperature physicists as well as for those working on the development of a theory of hyperbolic thermoelasticity with fractional order.
- According to these results, we have to construct a new classification to all materials according to their fractional parameter. This parameter becomes new indicator of their ability to conduct the thermal energy.

## References

1. M.A. BIOT, *Thermoelasticity and irreversible thermodynamics*, Journal of Applied Physics, **27**, 240–253, 1956.
2. H.W. LORD, Y. SHULMAN, *A generalized dynamical theory of thermoelasticity*, Journal of the Mechanics and Physics of Solids, **15**, 299–307, 1967.
3. A.E. GREEN, K.A. LINDSAY, *Thermoelasticity*, Journal of Elasticity, **2**, 1–7, 1972.
4. D.Y. TZOU, *A unified field theory for heat conduction from macro- to micro-scale*, ASME Journal of Heat Transfer, **117**, 8–16, 1995.
5. A.E. GREEN, P.M. NAGHDI, *A re-examination of the basic postulates of thermomechanics*, Proceedings of the Royal Society, **432**, 171–194, 1991.

6. A.E. GREEN, P.M. NAGHDI, *On undamped heat waves in an elastic solid*, Journal of Thermal Stresses, **15**, 253–264, 1992.
7. A.E. GREEN, P.M. NAGHDI, *Thermoelasticity without energy dissipation*, Journal of Elasticity, **31**, 189–209, 1993.
8. P.J. CHEN, M.E. GURTIN, *On a theory of heat conduction involving two temperatures*, Zeitschrift für angewandte Mathematik und Physik (ZAMP), **19**, 614–627, 1968.
9. P.J. CHEN, W.O. WILLIAMS, *A note on non-simple heat conduction*, Zeitschrift für angewandte Mathematik und Physik (ZAMP), **19**, 969–970, 1968.
10. P.J. CHEN, M.E. GURTIN, W.O. WILLIAMS, *On the thermodynamics of non-simple elastic materials with two temperatures*, Zeitschrift für angewandte Mathematik und Physik (ZAMP), **20**, 107–112, 1969.
11. B.A. BOLEY, I.S. TOLINS, *Transient coupled thermoelastic boundary value problems in the half-space*, ASME Journal of Applied Mechanics, **29**, 637–646, 1962.
12. W.E. WARREN, P.J. CHEN, *Wave propagation in the two temperature theory of thermoelasticity*, Acta Mechanica, **16**, 21–33, 1973.
13. H.M. YOUSSEF, *Dependence of the modulus of elasticity and the thermal conductivity on the reference temperature in generalized thermoelasticity for an infinite material with a spherical cavity*, Applied Mathematics and Mechanics (English edition), **26**, 470–475, 2005.
14. A.M. ZENKOUR, A.E. ABOUELREGAL, *The effect of two temperatures on a functionally graded nanobeam induced by a sinusoidal pulse heating*, Structural Engineering and Mechanics, **51**, 199–214, 2014.
15. A.M. ZENKOUR, A.E. ABOUELREGAL, *Effect of harmonically varying heat on FG nanobeams in the context of a nonlocal two-temperature thermoelasticity theory*, European Journal of Computational Mechanics, **23**, 1–14, 2014.
16. I.A. ABBAS, A.M. ZENKOUR, *Two-temperature generalized thermoelastic interaction in an infinite fiber-reinforced anisotropic plate containing a circular cavity with two relaxation times*, Journal of Computational and Theoretical Nanoscience, **11**, 1–7, 2014.
17. A.M. ZENKOUR, A.E. ABOUELREGAL, *State-space approach for an infinite medium with a spherical cavity based upon two-temperature generalized thermoelasticity theory and fractional heat conduction*, Zeitschrift für angewandte Mathematik und Physik (ZAMP), **65**, 149–164, 2014.
18. Y. SUN, D. FANG, M. SAKA, A.K. SOH, *Laser-induced vibrations of micro-beams under different boundary conditions*, International Journal of Solids and Structures, **45**, 1993–2013, 2008.
19. B.S. YILBAS, M. SAMI, S.Z. SHUJA, *Laser-induced thermal stresses on steel surface*, Optics and Lasers in Engineering, **30**, 25–37, 1998.
20. D.Y. TZOU, *Macro- to Microscale Heat Transfer: the Lagging Behavior*, Series in Chemical and Mechanical Engineering, Taylor & Francis, Washington, DC, 1997.
21. M. CAPUTO, *Linear model of dissipation whose  $Q$  is almost frequency independent-II*, Geophysical Journal of the Royal Astronomical Society, **13**, 529–539, 1967.
22. M. CAPUTO, F. MAINARDI, *A new dissipation model based on memory mechanism*, Pure and Applied Geophysics, **91**, 134–147, 1971.

23. Y.Z. POVSTENKO, *Fractional heat conduction equation and associated thermal stress*, Journal of Thermal Stresses, **28**, 83–102, 2005.
24. Y.Z. POVSTENKO, *Thermoelasticity that uses fractional heat conduction equation*, Journal of Mathematical Sciences, **162**, 296–305, 2009.
25. X. JIANG, M. XU, *The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate systems*, Physica A, **389**, 3368–3374, 2010.
26. A.E. ABOUELREGAL, A.M. ZENKOUR, *Effect of fractional thermoelasticity on a two-dimensional problem of a mode I crack in a rotating fibre-reinforced thermoelastic medium*, Chinese Physics B, **22**, 22, 10 (108102 – 8 pp.), 2013.
27. I.A. ABBAS, A.M. ZENKOUR, *Semi-analytical and numerical solution of fractional order generalized thermoelastic in a semi-infinite medium*, Journal of Computational and Theoretical Nanoscience, **11**, 1592–1596, 2014.
28. K.S. MILLER, B. ROSS, *An Introduction to the Fractional Integrals and Derivatives, Theory and Applications*, John Wiley and Sons Inc., New York, 1993.
29. I. PODLUBNY, *Fractional Differential Equations*, Academic Press, New York, 1999.
30. M.A. EZZAT, *Magneto-thermoelasticity with thermoelectric properties and fractional derivative heat transfer*, Physica B: Physics of Condensed Matter, **406**, 30–35, 2011.
31. G. JUMARIE, *Derivation and solutions of some fractional Black-Scholes equations in coarse-grained space and time. Application to Merton's optimal portfolio*, Computers & Mathematics with Applications, **59**, 1142–1164, 2010.
32. G. HONIG, U. HIRDES, *A method for the numerical inversion of the Laplace transform*, Journal of Computational and Applied Mathematics, **10**, 113–132, 1984.

Received May 2, 2014; revised version August 13, 2014.

---