Some considerations of fundamental solution in micropolar thermoelastic materials with double porosity

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Abstract

This paper is concerned with micropolar thermoelastic materials with double porosity structure. The system of the equations of the assumed model is based on the equations of motion, equilibrated stress equations of motion and heat conduction equation for material with double porosity. The explicit expressions for the fundamental solution of the system of equations in case of steady vibrations are presented. The desired solutions are constructed by the use of elementary functions. Some basic properties are also established. The aspect of the particular cases of Scarpetta et al.[25], Scarpetta [42], Ciarlette et al [45] and Svanadze [51] are also deduced form the present investigation.

Keywords: Micropolar thermoelasticity; Double porosity; Fundamental solution; Steady vibrations

1. Introduction

Porous media theories play an important role in many branches of engineering including material science, the petroleum industry, chemical engineering, biomechanics and other such fields of engineering. The construction and the intensive investigation of the theories of continua with microstructures arise by the wide use of porous materials into engineering and technology. Representation of a fluid saturated porous medium as a single phase material has been virtually discarded. The material with the pore spaces such as concrete can be treated easily because all concrete ingredients have the same motion if the concrete body is deformed. However the situation is more complicated if the pores are filled with liquid and in that case the solid and liquid phases have different motions. Due to these different motions, the different material properties and the complicated geometry of pore structures, the mechanical behavior of a fluid saturated porous thermoelastic medium becomes very difficult. So researchers from time to time, have tried to overcome this difficulty and we see many porous media in the literature. A brief historical background of these theories is given by de Boer [1,2].

As far as modern era is concerned Biot [3] proposed a general theory of three-dimensional deformation of fluid saturated porous salts. Biot theory is based on the assumption of compressible constituents and till recently, some of his results have been taken as standard references and basis for subsequent analysis in acoustic, geophysics and other such fields. Another interesting theory is given by Bowen [4], de Boer and Ehlers [5] in which all the constituents of a porous medium are assumed to be incompressible. The fluid saturated porous material is modeled as a two phase system composed of an incompressible solid phase and incompressible fluid phase, thus meeting the many problems in engineering practice, e.g. in soil mechanics. One important generalization of Biot's theory of poroelasticity that has been studied extensively started with the works by Barenblatt et al. [6], where the double porosity model was first proposed to express the fluid flow in hydrocarbon reservoirs and aquifers.

The double porosity model represents a new possibility for the study of important problems concerning the civil engineering. It is well-known that, under super- saturation conditions due to water of other fluid effects, the so called neutral pressures generate unbearable stress states on the solid matrix and on the fracture faces, with severe (sometimes disastrous) instability effects like landslides, rock fall or soil fluidization (typical phenomenon connected with propagation of seismic waves). In such a context it seems possible, acting suitably on the boundary pressure state, to regulate the internal pressures in order to deactivate the noxious effects related to neutral pressures; finally, a further but connected positive effect could be lightening of the solid matrix/fluid system.

Wilson and Aifanits [7] presented the theory of consolidation with the double porosity. Khaled, Beskos and Aifantis [8] employed a finite element method to consider the numerical solutions of the differential equation of the theory of consolidation with double porosity developed by Aifantis[7]. Wilson and Aifantis [9] discussed the propagation of acoustics waves in a fluid saturated porous medium. The propagation of acoustic waves in a fluid-saturated porous medium containing a continuously distributed system of fractures is discussed. The porous medium is assumed to consist of two degrees of porosity and the resulting model thus yields three types of longitudinal waves, one associated with the elastic properties of the matrix material and one each for the fluids in the pore space and the fracture space.

Beskos and Aifantis [10] presented the theory of consolidation with double porosity-II and obtained the analytical solutions to two boundary value problems. Khalili and Valliappan [11] studied the unified theory of flow and deformation in double porous media. Aifantis [12-15] introduced a multi-porous system and studied the mechanics of diffusion in solids. Moutsopoulos et al. [16] obtained the numerical simulation of transport phenomena by using the double porosity/ diffusivity continuum model. Khalili and Selvadurai [17] presented a fully coupled constitutive model for thermo-hydro –mechanical analysis in elastic media with double porosity structure. Pride and Berryman [18] studied the linear dynamics of double –porosity dual-permeability materials. Straughan [19] studied the stability and uniqueness in double porous elastic media .

Svanadze [20-24] investigated some problems on elastic solids, viscoelastic solids and thermoelastic solids with double porosity. Scarpetta et al. [25, 26] proved the uniqueness theorems in the theory of thermoelasticity for solids with double porosity and also obtained the fundamental solutions in the theory of thermoelasticity for solids with double porosity.

Nunziato and Cowin [27] developed a nonlinear theory of elastic material with voids. Later, cowin and nunziato [28] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behavior of porous solids. They also considered several applications of the linear theory by investigating the response of the materials to homogeneous deformations, pure bending of beams and small amplitudes of acoustic waves. Nunziato and Cowin have established a theory for the behavior of porous solids in which the skeletal or matrix materials are elastic and the interstices are voids of material.

Iesan and Quintanilla [29] used the Nunziato-Cowin theory of materials with voids to derive a theory of thermoelastic solids, which have a double porosity structure. This theory is not based on Darcy's law. In contrast with the classical theory of elastic materials with the double porosity, the double porosity structure in the case of equilibrium is influenced by the displacement field. Marin et al. [56] presented a new model for micropolar bodies with double porosity.

The mechanical behavior of solids with voids, solids containing microscopic components, cannot be described by means of the classical theory of elasticity. In reality, almost all materials possess microstructure and in such materials, microstructural motions cannot be ignored. Eringen [30] introduced the theory of micropolar elasticity which has aroused much interest in recent years because of its possible utility in investigating the deformation properties of solids for which the classical theory in adequate. The micropolar theory has been useful in investigating material consisting of bar like molecules, which exhibit the microrotational effects and can support body and surface couples. A micropolar continuum is a collection of interconnected particles in the form of small rigid bodies undergoing both translational and rotational motions. The force at a point of the surface element of bodies is completely characterized by stress vector and couple stress vector at that point.

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua thermal effect. The comprehensive review on this theory was given by Eringen[31] and Nowacki [32]. Touchert et al. [33] derived the basic equations of the linear theory of micropolar coulpled thermoelasticity. Chandrasekharaiah [34] developed a heat flux dependent micropolar thermoelasticity. Boschi and Iesan [35] extended a generalized theory of micropolar thermoelasticity that permits the transmission of heat as thermal waves at finite speed.

The construction of fundamental solutions has great importance in many mathematical, physical and engineering problems.To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties respectively. Hetnarski [36,37]studied the fundamental solutions in the classical theory of coupled thermoelasticity. The information related to fundamental solutions of differential equations is contained in the books of Hörmander [38,39].Various authors [41-54] have derived the fundamental solutions in different theories of continuum mechanics.

In the present paper, the fundamental solution of system of equations in the case of steady vibrations in terms of elementary functions are constructed and basic properties of the fundamental solution are established. Some particular cases of interest have also been deduced.

2. Basic equations

Let $\mathbf{x}=(x_1,x_2,x_3)$ be the point of the Euclidean three-dimensional space $R^3, |x|=(\mathrm{x}_1^2+\mathrm{x}_2^2+\mathrm{x}_3^2)^{\frac{1}{2}}, \mathbf{D}_x=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$, and let t denote the time variable.

Following Marin et al. [56], the basic equations for isotropic, homogeneous micropolar thermoelastic material with double porosity structure in the absence of body forces, body couples, extrinsic equilibrated body forces and heat sources are:

$$
\begin{aligned}
&(\mu+\kappa)\Delta\bar{\mathbf{u}}+(\lambda+\mu)\,\mathrm{grad\,div}\,\bar{\mathbf{u}}+\kappa\,\mathrm{curl}\bar{\mathbf{\Phi}}+b\,\mathrm{grad}\bar{\varphi}+d\,\mathrm{grad}\bar{\psi}-\beta\,\mathrm{grad}\bar{T}=\rho\ddot{\bar{u}}\\
&(\gamma\Delta-2\kappa)\bar{\mathbf{\Phi}}+(\alpha+\beta)\,\mathrm{grad\,div}\,\bar{\mathbf{\Phi}}+\kappa\,\mathrm{curl}\bar{\mathbf{u}}+c_0\mathrm{grad}\bar{\varphi}+d_0\mathrm{grad}\bar{\psi}=\rho_j\ddot{\bar{\mathbf{\Phi}}}\\
&\alpha\Delta\bar{\varphi}+b_1\Delta\bar{\psi}-b\,\mathrm{div}\,\bar{\mathbf{u}}-\alpha_1\bar{\varphi}-\alpha_3\bar{\psi}+\gamma_1\bar{T}-c_0\mathrm{div}\,\bar{\mathbf{\Phi}}=\kappa_1\ddot{\bar{\varphi}}\\
&b_1\Delta\bar{\varphi}+\gamma_0\Delta\bar{\psi}-d\,\mathrm{div}\,\bar{\mathbf{u}}-\alpha_3\bar{\varphi}-\alpha_2\bar{\psi}+\gamma_2\bar{T}-d_0\mathrm{div}\,\bar{\mathbf{\Phi}}=\kappa_2\ddot{\bar{\psi}}\\
&K^*\Delta\bar{T}-\beta T_0\mathrm{div}\,\dot{\bar{\mathbf{u}}}-\gamma_1T_0\dot{\bar{\varphi}}-\gamma_2T_0\dot{\bar{\psi}}=\rho C^*\dot{\bar{T}}
\end{aligned}
\tag{1}
$$

where $\bar{\mathbf{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3)$ is the displacement vector; $\bar{\mathbf{\Phi}}=(\bar{\Phi}_1,\bar{\Phi}_2,\bar{\Phi}_3)$ is the microrotation vector, λ and μ are Lame's constants; ρ is the mass density; ρ_j is cofficient of inertia; $\beta=(3\lambda+2\mu)\alpha_t$; α_t is the linear thermal expansion; C^* is the specific heat at constant strain; $\bar{u}_i$ is the displacement components; κ_1 and κ_2 are coefficients of equilibrated inertia; φ' and ψ' are the volume fraction fields corresponding to pores and fissures respectively ; K^* is the coefficient of thermal conductivity and $b,d,b_1,\gamma_0,\gamma_1,\gamma_2,c_0,d_0,\kappa$ are constitutive coefficients; δ_{ij} is the Kronecker's delta; $\bar{T}$ is the temperature change measured form the absolute temperature $T_0\,(T_0\neq 0)$; a superposed dot represents differentiation with respect to time variable t and Δ is the Laplacian operator.

If the displacement vector $\bar{\mathbf{u}}$, microrotation vector $\bar{\mathbf{\Phi}}$,volume fractions fields $\bar{\varphi},\bar{\psi}$ and temperature distribution $\bar{T}$ have a harmonic time variation as

$$
\left\{\bar{\mathbf{u}},\bar{\mathbf{\Phi}},\bar{\varphi},\bar{\psi},\bar{T}\right\}(\mathbf{x},\mathrm{t})=\mathrm{Re}\left[\{\mathbf{u},\mathbf{\Phi},\varphi,\psi,T\}(\mathbf{x})\,e^{-i\omega t}\right]
\tag{2}
$$

Using (2) in (1) yield the system of steady vibrations as

$$
\begin{aligned}
&\left[\left(\mu+\kappa\right)\Delta+\rho\omega^2\right]\mathbf{u}+\left(\lambda+\mu\right)\operatorname{grad}\operatorname{div}\mathbf{u}+\kappa\operatorname{curl}\mathbf{\Phi}+b\operatorname{grad}\varphi+d\operatorname{grad}\psi-\beta\operatorname{grad}T=0\\
&\left(\gamma\Delta+\mu_1\right)\mathbf{\Phi}+\left(\alpha+\beta\right)\operatorname{grad}\operatorname{div}\mathbf{\Phi}+\kappa\operatorname{curl}\mathbf{u}+c_0\operatorname{grad}\varphi+d_0\operatorname{grad}\psi=0\\
&\left(\alpha\Delta+\mu_2\right)\varphi+\left(b_1\Delta-\alpha_3\right)\psi-b\operatorname{div}\mathbf{u}+\gamma_1 T-c_0\operatorname{div}\mathbf{\Phi}=0\\
&\left(b_1\Delta-\alpha_3\right)\varphi+\left(\gamma_0\Delta+\mu_3\right)\psi-d\operatorname{div}\mathbf{u}+\gamma_2 T-d_0\operatorname{div}\mathbf{\Phi}=0\\
&(k_3\Delta-\rho C^*)T-\beta T_0\operatorname{div}\mathbf{u}-\gamma_1 T_0\varphi-\gamma_2 T_0\psi=0
\end{aligned}
\qquad (3)
$$

where ω is the oscillation frequency $\left(\omega>0\right)$, and

$$\mu_1=\rho_j\omega^2-2\kappa,\ \ \mu_2=\kappa_1\omega^2-\alpha_1,\ \ \mu_3=\kappa_2\omega^2-\alpha_2,\ \ k_3=-\frac{K^*}{i\omega}$$

Introducing the matrix differential operator

$$\mathbf{E}(\mathbf{D}_x)=\left\|E_{gh}(\mathbf{D}_x)\right\|_{9\times 9}$$

where

$$E(\mathbf{D}_x)=\left[\left(\mu+k\right)\Delta+\rho\omega^2\right]\delta_{mn}+\left(\lambda+\mu\right)\frac{\partial^2}{\partial x_m\partial x_n},$$

$$E_{m,n+3}(\mathbf{D}_x)=E_{m+3,n}(\mathbf{D}_x)=\kappa\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$E_{m7}(\mathbf{D}_x)=-E_{7m}(\mathbf{D}_x)=b\frac{\partial}{\partial x_m},\ \ E_{m8}(\mathbf{D}_x)=-E_{8m}(\mathbf{D}_x)=d\frac{\partial}{\partial x_m},$$

$$E_{m9}(\mathbf{D}_x)=-\beta\frac{\partial}{\partial x_m},\ \ E_{m+3,n+3}(\mathbf{D}_x)=\left(\gamma\Delta+\mu_1\right)\delta_{mn}+\left(\alpha+\beta\right)\frac{\partial^2}{\partial x_m\partial x_n},$$

$$E_{m+3,7}(\mathbf{D}_x)=-E_{7,n+3}(\mathbf{D}_x)=c_0\frac{\partial}{\partial x_m},\ E_{m+3,8}(\mathbf{D}_x)=-E_{8,n+3}(\mathbf{D}_x)=d_0\frac{\partial}{\partial x_m},$$

$$E_{77}(\mathbf{D}_x)=\alpha\Delta+\mu_2\,,E_{78}(\mathbf{D}_x)=E_{87}(\mathbf{D}_x)=b_1\Delta-\alpha_3\,,E_{79}(\mathbf{D}_x)=\gamma_1\,,$$

$$E_{89}(\mathbf{D}_x)=\gamma_2\,,E_{97}(\mathbf{D}_x)=-\gamma_1 T_0\,,E_{98}(\mathbf{D}_x)=-\gamma_2 T_0\,,E_{88}(\mathbf{D}_x)=\gamma_0\Delta+\mu_3,$$

$$E_{9m}(\mathbf{D}_x)=-\beta T_0\frac{\partial}{\partial x_m},E_{m+3,9}(\mathbf{D}_x)=0=E_{9,n+3}(\mathbf{D}_x),$$

$$E_{99}(\mathbf{D}_x)=k_3\Delta-\rho C^* \qquad\qquad m,n=1,2,3$$

δ_{mn} is the Kronecker's delta and ε_{mrn} is the alternating symbol.

The system (3) can be written as

$$\mathbf{E}(\mathbf{D}_x)\mathbf{U}(\mathbf{x})=\mathbf{0}$$

where $\mathbf{U}=(\mathbf{u},\mathbf{\Phi},\varphi,\psi,\mathrm{T})$ is a nine-component vector function on R^3.

We assume that

$$\alpha_4\alpha_5\alpha_6 k_3\gamma\left(\mu+\kappa\right)\neq 0 \tag{4}$$

where

$\alpha_4=\lambda+2\mu+\kappa, \alpha_5=\alpha+\beta+\gamma, \alpha_6=\alpha\gamma_0-b_1^2$. Evidently, if condition (4) are satisfied, then $\boldsymbol{E}$ is the elliptic differential operator [38].

Definition: The fundamental solution of the system (3) (the fundamental matrix of operator $\boldsymbol{E}$) is the matrix $\Lambda(\mathrm{x})=\left\|\Lambda_{gh}(x)\right\|_{9\times 9}$ satisfying condition [38]

$$\boldsymbol{E}(\mathbf{D}_x)\Lambda(\mathrm{x})=\delta(\mathrm{x})\mathbf{I}(\mathrm{x}) \tag{5}$$

where δ is the Dirac delta, $\boldsymbol{I}=\left\|\delta_{gh}\right\|_{9\times 9}$ is the unit matrix, and $\mathbf{x}\in R^3$.

Now we construct the matrix $\mathbf{\Lambda}(\mathrm{x})$ in terms of elementary functions and also establish some basic properties.

3. Fundamental solution of the system of equations of steady vibrations

We consider the system of equations

$$\begin{aligned}
&\left[\left(\mu+\kappa\right)\Delta+\rho\omega^2\right]\mathbf{u}+\left(\lambda+\mu\right)\mathrm{grad\,div}\,\mathbf{u}+\kappa\mathrm{curl}\mathbf{\Phi}+b\mathrm{grad}\varphi+d\mathrm{grad}\psi-\beta\mathrm{grad}T=\mathbf{F}'\\
&\left(\gamma\Delta+\mu_1\right)\mathbf{\Phi}+\left(\alpha+\beta\right)\mathrm{grad\,div}\,\mathbf{\Phi}+\kappa\mathrm{curl}\mathbf{u}+c_0\mathrm{grad}\varphi+d_0\mathrm{grad}\psi=\mathbf{F}''\\
&\left(\alpha\Delta+\mu_2\right)\varphi+\left(b_1\Delta-\alpha_3\right)\psi-b\,\mathrm{div}\,\mathbf{u}+\gamma_1 T-c_0\mathrm{div}\,\mathbf{\Phi}=f'\\
&\left(b_1\Delta-\alpha_3\right)\varphi+\left(\gamma_0\Delta+\mu_3\right)\psi-d\,\mathrm{div}\,\mathbf{u}+\gamma_2 T-d_0\mathrm{div}\,\mathbf{\Phi}=f''\\
&(k_3\Delta-\rho C^*)T-\beta T_0\mathrm{div}\,\mathbf{u}-\gamma_1 T_0\varphi-\gamma_2 T_0\psi=f'''
\end{aligned} \tag{6}$$

where $\mathbf{F}'$ and $\mathbf{F}''$ are three-component vector functions on $\mathbf{R}^3$; f', f'' and f''' are scalar functions on R^3.

The system (6) may be written in the form

$$\mathbf{E}^{tr}(\mathbf{D}_x)\mathbf{U}(\mathrm{x})=\boldsymbol{Q}(\mathrm{x}) \tag{7}$$

where $\boldsymbol{E}^{tr}$ is the transpose of matrix $\boldsymbol{E}$, $\boldsymbol{Q} = (\mathbf{F}^{'}, \mathbf{F}^{''}, f^{'}, f^{''}, f^{'''})$ is the nine-component vector function on R^3, and $\mathbf{x} \in R^3$.

Applying the operator div to first and second equations of system (6), we get

$$\begin{aligned}
&\left[\alpha_4\Delta + \rho\omega^2\right]\operatorname{div}\mathbf{u} + b\Delta\varphi + d\Delta\psi - \beta\Delta T = \operatorname{div}\mathbf{F}^{'} \\
&\left(\alpha_5\Delta + \mu_1\right)\operatorname{div}\boldsymbol{\Phi} + c_0\Delta\varphi + d_0\Delta\psi = \operatorname{div}\mathbf{F}^{''} \\
&\left(\alpha\Delta + \mu_2\right)\varphi + \left(b_1\Delta - \alpha_3\right)\psi - b\operatorname{div}\mathbf{u} + \gamma_1 T - c_0\operatorname{div}\boldsymbol{\Phi} = f^{'} \\
&\left(b_1\Delta - \alpha_3\right)\varphi + \left(\gamma_0\Delta + \mu_3\right)\psi - d\operatorname{div}\mathbf{u} + \gamma_2 T - d_0\operatorname{div}\boldsymbol{\Phi} = f^{''} \\
&(k_3\Delta - \rho C^*)T - \beta T_0\operatorname{div}\mathbf{u} - \gamma_1 T_0\varphi - \gamma_2 T_0\psi = f^{'''}
\end{aligned} \tag{8}$$

The system (8) can be written as

$$\mathbf{H}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}} \tag{9}$$

where $\mathbf{S} = (\operatorname{div}\mathbf{u}, \operatorname{div}\boldsymbol{\Phi}, \varphi, \psi, \mathrm{T}), \tilde{\mathbf{Q}} = (\operatorname{div}\mathbf{F}^{'}, \operatorname{div}\mathbf{F}^{''}, f^{'}, f^{''}, f^{'''}) = (f_1, f_2, f_3, f_4, f_5)$ and

$$\mathbf{H}(\Delta) = \left\| H_{mn}(\Delta) \right\|_{5\times5} = \begin{Vmatrix} \alpha_4\Delta + \rho\omega^2 & 0 & b\Delta & d\Delta & -\beta\Delta \\ 0 & \alpha_5\Delta + \mu_1 & c_0\Delta & d_0\Delta & 0 \\ -b & -c_0 & \alpha\Delta + \mu_2 & b_1\Delta - \alpha_3 & \gamma_1 \\ -d & -d_0 & b_1\Delta - \alpha_3 & \gamma_0\Delta + \mu_3 & \gamma_2 \\ -\beta T_0 & 0 & -\gamma_1 T_0 & -\gamma_2 T_0 & k_3\Delta - \rho C^* \end{Vmatrix}_{5\times5} \tag{10}$$

The system (8) may be written as

$$\Gamma_1(\Delta)\mathbf{S} = \boldsymbol{\Psi} \tag{11}$$

where

$$\begin{aligned}
&\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) \quad \Psi_n = g_1 \sum_{m=1}^{5} H_{mn}^{*} f_m \\
&\Gamma_1(\Delta) = g_1 \det\mathbf{H}(\Delta) \quad g_1 = \frac{1}{\alpha_4\alpha_5\alpha_6 k_3} \quad n = 1,2,3,4,5
\end{aligned} \tag{12}$$

where H_{mn}^{*} is the cofactor of the element H_{mn} of the matrix **H.**

From (10) and (11), we see that

$$\Gamma_1(\Delta) = \prod_{m=1}^{5}(\Delta + \xi_m^2)$$

where $\xi_m^2, \;\; m = 1,2,3,4,5$ are the roots of the equation $\Gamma_1(-\chi) = 0$ (with respect to χ).

Applying the operator $\left(\gamma\Delta + \mu_1\right)$ and $\kappa\,\mathrm{curl}$ to Eqs. $(6)_1$ and $(6)_2$, respectively, we get

$$(\gamma\Delta + \mu_1)\left[\left(\mu + \kappa\right)\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \rho\omega^2\mathbf{u}\right] + \kappa(\gamma\Delta + \mu_1)\,\mathrm{curl}\,\mathbf{\Phi}$$
$$= (\gamma\Delta + \mu_1)\left[F^{'} - b\,\mathrm{grad}\varphi - d\;\mathrm{grad}\psi + \beta\mathrm{grad}T\right] \tag{13}$$

$$\kappa(\gamma\Delta + \mu_1)\,\mathrm{curl}\,\mathbf{\Phi} = -\kappa^2\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} + \kappa\,\mathrm{curl}\mathbf{F}^{"} \tag{14}$$

On using (14) and equality

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathrm{grad}\,\mathrm{div}\,\mathbf{u}\text{-}\Delta\mathbf{u} \tag{15}$$

in Eq. (13), we obtain

$$\left[\left\{(\gamma\Delta + \mu_1)\left(\mu + \kappa\right) + \kappa^2\right\}\Delta + \rho\omega^2(\gamma\Delta + \mu_1)\right]\mathbf{u} + \left[(\lambda + \mu)(\gamma\Delta + \mu_1) - \kappa^2\right]\mathrm{grad}\,\mathrm{div}\mathbf{u}$$
$$= (\gamma\Delta + \mu_1)\left[F^{'} - b\,\mathrm{grad}\varphi - d\;\mathrm{grad}\psi + \beta\mathrm{grad}T\right] - \kappa\,\mathrm{curl}\,\mathbf{F}^{"} \tag{16}$$

Applying the operator $\Gamma_1(\Delta)$ to Eq. (16) and after using Eq. (11), we obtain

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{u} = \mathbf{\Psi}^{'} \tag{17}$$

where

$$\Gamma_2(\Delta) = g_2\left[\gamma(\mu+\kappa)\Delta^2 + (\mu\mu_1 + \mu_1\kappa + \kappa^2 + \rho\omega^2\gamma)\Delta + \rho\omega^2\mu_1\right]$$

$$g_2 = \frac{1}{\gamma(\mu+\kappa)}$$

and

$$\mathbf{\Psi}^{'} = g_2[(\gamma\Delta + \mu_1)(\Lambda_1\mathbf{F}^{'} - b\,\mathrm{grad}\Psi_3 - d\;\mathrm{grad}\Psi_4 + \beta\mathrm{grad}\Psi_5) - \Lambda_1\mathrm{curl}\mathbf{F}^{"}$$
$$-\left\{(\lambda+\mu)(\gamma\Delta + \mu_1) - \kappa^2\right\}\kappa\mathrm{grad}\Psi_1] \tag{18}$$

We see that

$$\Gamma_2(\Delta)=(\Delta+\xi_6^2)(\Delta+\xi_7^2) \tag{19}$$

where ξ_6^2 and ξ_7^2 are the roots of the equation (with respect to χ).

$$\gamma(\mu+\kappa)\chi^2+(\mu\mu_1+\mu_1\kappa+\kappa^2+\rho\omega^2\gamma)\chi+\rho\omega^2\mu_1=0$$

Similarly, from Eqs. $(6)_1,(6)_2$ and (11), we obtain

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{\Phi}=\mathbf{\Psi}^{"} \tag{20}$$

where

$$\mathbf{\Psi}^{"}=g_2\Lambda_1(\Delta)[-\kappa\,\mathrm{curl}\,\mathbf{F}^{'}+\{(\mu+\kappa)\Delta+\rho\omega^2\}\mathbf{F}^{"}-g_2[(\alpha+\beta)\{(\mu+\kappa)\Delta+\rho\omega^2\}-\kappa^2]\,\mathrm{grad}\,\Phi_2$$
$$-c_0g_2[(\mu+\kappa)\Delta+\rho\omega^2]\,\mathrm{grad}\,\Phi_3-d_0g_2[(\mu+\kappa)\Delta+\rho\omega^2]\,\mathrm{grad}\,\Phi_4 \tag{21}$$

From Eqs. (17), (20) and (11), we obtain

$$\mathbf{\Gamma}(\Delta)\mathbf{U}(\mathrm{x})=\tilde{\mathbf{\Psi}}(\mathrm{x}) \tag{22}$$

where $\mathbf{\Psi}=(\mathbf{\Psi}^{'},\mathbf{\Psi}^{"},\Psi_3,\Psi_4,\Psi_5)$ and

$$\Gamma(\Delta)=\left\|\Gamma_{ij}(\Delta)\right\|_{9\times 9}$$
$$\Gamma_{pp}(\Delta)=\Gamma_1(\Delta)\Gamma_2(\Delta)=\prod_{m=1}^{7}(\Delta+\xi_m^2)$$
$$\Gamma_{nn}(\Delta)=\Gamma_1(\Delta)\qquad \Gamma_{ij}(\Delta)=0$$

$$p=1,2,3,\ldots,6\quad n=7,8,9\quad i,j=1,2,\ldots,9\quad i\neq j$$

In what follows we take

$$r_{l1}(\Delta)=g_1g_2[(\gamma\Delta+\mu_1)(-bH_{l3}^*-dH_{l4}^*+\beta H_{l5}^*)-\{(\lambda+\mu)(\gamma\Delta+\mu_1)-\kappa^2\}H_{l1}^*]$$
$$r_{l2}(\Delta)=-c_0g_1g_2[(\mu+\kappa)\Delta+\rho\omega^2]H_{l3}^*-d_0g_1g_2[(\mu+\kappa)\Delta+\rho\omega^2]H_{l4}^*$$
$$-g_1g_2[(\alpha+\beta)\{(\mu+\kappa)\Delta+\rho\omega^2\}-\kappa^2]H_{l2}^* \tag{23}$$
$$r_{lj}(\Delta)=g_1H_{lj}^*\quad l=1,2,3,4,5\quad j=3,4,5$$

It is evident that $r_{12}(\Delta)=r_{21}(\Delta)$. From Eqs. (18) and (20) ,by virtue of Eqs.(12) and (23), we have

$$\boldsymbol{\Psi}^{'} = [g_2(\gamma\Delta + \mu_1)\Gamma_1\mathbf{I} + r_{11}\,\text{grad div}]\mathbf{F}^{'} + [-\kappa g_2\Gamma_1\text{curl} + r_{21}\,\text{grad div}]\mathbf{F}^{''}$$
$$+ r_{31}\,\text{grad}\, f^{'} + r_{41}\,\text{grad}\, f^{''} + r_{51}\,\text{grad}\, f^{'''}$$

$$\boldsymbol{\Psi}^{''} = [-\kappa b_2\Gamma_1\text{curl} + r_{12}\,\text{grad div}]\mathbf{F}^{'} + [g_2\{(\mu+\kappa)\Delta + \rho\omega^2\}\Gamma_1\mathbf{I} + r_{22}\,\text{grad div}]\mathbf{F}^{''}$$
$$+ r_{32}\,\text{grad}\, f^{'} + r_{42}\,\text{grad}\, f^{''} + r_{52}\,\text{grad}\, f^{'''}$$

$$\Psi_j = r_{1j}\text{div}\,\mathbf{F}^{'} + r_{2j}\text{div}\,\mathbf{F}^{''} + r_{3j}\, f^{'} + r_{4j}\, f^{''} + r_{5j}\, f^{''} \quad j = 3,4,5 \tag{24}$$

where $\mathbf{I} = \left\|\delta_{ej}\right\|_{3\times 3}$ is the unit matrix.

Therefore, from Eq. (23),we have

$$\tilde{\Psi}(\text{x}) = \mathbf{N}^{tr}(\mathbf{D}_\text{x})\mathbf{Q}(\text{x}) \tag{25}$$

where

$$\mathbf{N}(\mathbf{D}_x) = \left\|N_{gh}(\mathbf{D}_x)\right\|_{9\times 9}$$
$$N_{mn}(\mathbf{D}_x) = g_2(\gamma\Delta + \mu_1)\Gamma_1(\Delta)\delta_{mn} + r_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}$$
$$N_{m,n+3}(\mathbf{D}_x) = N_{m+3,n}(\mathbf{D}_x) = -\kappa g_2\Gamma_1(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r} + r_{12}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}$$
$$N_{mp}(\mathbf{D}_x) = r_{1,p-4}(\Delta)\frac{\partial}{\partial x_m}$$
$$N_{m+3,n+3}(\mathbf{D}_x) = g_2\{(\mu+\kappa)\Delta + \rho\omega^2\}\Gamma_1(\Delta)\delta_{mn} + r_{22}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}$$

$$N_{m+3,p}(\mathbf{D}_x) = r_{2,p-4}(\Delta)\frac{\partial}{\partial x_m} \quad N_{pm}(\mathbf{D}_x) = r_{p-4,1}\frac{\partial}{\partial x_m}$$
$$N_{p,m+3}(\mathbf{D}_x) = r_{p-4,2}(\Delta)\frac{\partial}{\partial x_m} \tag{26}$$
$$N_{pq}(\mathbf{D}_x) = r_{p-4,q-4}(\Delta)$$
$$m,n = 1,2,3 \quad p,q = 7,8,9$$

In view of Eqs. (7) and (25), from Eq. (22) ,it is found that $\boldsymbol{\Gamma}\mathbf{U}=\mathbf{N}^{tr}\mathbf{E}^{tr}\mathbf{U}$. It is evident that

$\mathbf{N}^{tr}\mathbf{E}^{tr} = \Gamma$ and hence

$$\mathbf{E}(\mathbf{D}_x)\,\mathbf{N}(\mathbf{D}_x) = \Gamma(\Delta) \tag{27}$$

We assume that

$$\xi_m^2 \neq \xi_n^2 \neq 0, \qquad m,n=1,2,...,7 \text{ and } m\neq n.$$

Let

$$\mathbf{Z}(\mathrm{x}) = \left\| Z_{ej}(\mathbf{D}_x) \right\|_{9\times 9}, \; Z_{mm}(\mathrm{x}) = \sum_{n=1}^{6} s_{1n}\varsigma_n(\mathrm{x}),$$
$$Z_{m+3,m+3}(\mathrm{x}) = \sum_{n=5}^{7} s_{2n}\varsigma_n(\mathrm{x}),$$
$$Z_{77}(\mathrm{x}) = Z_{88}(\mathrm{x}) = Z_{99}(\mathrm{x}) = \sum_{n=1}^{4} s_{3n}\varsigma_n(\mathrm{x}),$$
$$Z_{ej}(\mathrm{x}) = 0, \; m=1,2,3 \qquad e,j=1,2,...,9 \quad e\neq j$$

where

$$\varsigma_n(\mathrm{x}) = -\frac{1}{4\pi|\mathrm{x}|} e^{i\xi_n|\mathrm{x}|}$$
$$s_{1l} = \prod_{\substack{m=1\\ m\neq l}}^{6} (\xi_m^2 - \xi_l^2)^{-1}, \quad l=1,2,3,4,5,6$$
$$s_{2e} = \prod_{\substack{m=5\\ m\neq e}}^{7} (\xi_m^2 - \xi_e^2)^{-1}, \quad e=5,6,7 \qquad (28)$$
$$s_{3j} = \prod_{\substack{m=1\\ m\neq j}}^{4} (\xi_m^2 - \xi_j^2)^{-1}, \quad j=1,2,3,4$$

Therefore, the matrix $\mathbf{Z}$ is the fundamental matrix of operator $\Gamma(\Delta)$, that is

$$\Gamma(\Delta)\mathbf{Z}(\mathrm{x}) = \delta(\mathrm{x})\mathbf{I}(\mathrm{x}) \qquad (29)$$

Introducing the matrix

$$\Lambda(\mathrm{x}) = \mathbf{N}(\mathbf{D}_x)\mathbf{Z}(\mathrm{x}) \qquad (30)$$

On using (29), in Eqs. (27) and (30), we obtain

$$\mathbf{E}(\mathbf{D}_x)\Lambda(\mathrm{x}) = \mathbf{E}(\mathbf{D}_x)\mathbf{N}(\mathbf{D}_x)\mathbf{Z}(\mathrm{x}) = \Gamma(\Delta)\mathbf{Z}(\mathrm{x}) = \delta(\mathrm{x})\mathbf{I}(\mathrm{x}) \qquad (31)$$

Hence, $\Lambda(\mathrm{x})$ is the solution of Eq.(5).

We will prove the following theorem.

Theorem 1. The matrix $\Lambda(\mathrm{x})$ defined by Eq. (31) is the fundamental solution of system (3).

Remark. The fundamental solution $\Lambda(\mathrm{x})$ of system (3) is constructed for $\xi_m \neq \xi_n \neq 0$ $(m, n = 1, 2, ..., 7 \text{ and } m \neq n)$. Evidently, by the above method, it is possible to construct the fundamental solution of system (3) for the cases where $\xi_m = 0$ and $\xi_m = \xi_n$.

4. Basic properties of the matrix $\Lambda(\mathrm{x})$

Corollary 1. Each column of the matrix $\Lambda(\mathrm{x})$ is the solution of the system (3) at every point $\mathbf{x} \in \mathbf{R}^3$ except the origin.

Corollary 2. If conditions (4) are satisfied, then the fundamental solution of the system

$$\begin{aligned}
&(\mu+\kappa)\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u} = 0\\
&\gamma\Delta\mathbf{\Phi} + (\alpha+\beta)\,\mathrm{grad\,div}\,\mathbf{\Phi} = 0\\
&\alpha\Delta\varphi + b_1\Delta\psi = 0\\
&b_1\Delta\varphi + \gamma_0\Delta\psi = 0\\
&k_3\Delta T = 0
\end{aligned} \tag{32}$$

is the matrix

$$\mathbf{\Omega}(\mathrm{x}) = \left\|\Omega_{mn}(\mathrm{x})\right\|_{9\times 9}$$

where

$$\begin{aligned}
&\mathbf{\Omega}_{lj}(\mathrm{x}) = \left(\frac{1}{\alpha_4}\mathrm{grad\,div} - \frac{1}{\mu+\kappa}\mathrm{curl\,curl}\right)\lambda_1(\mathrm{x}),\\
&\mathbf{\Omega}_{lm}(\mathrm{x}) = \mathbf{\Omega}_{ml}(\mathrm{x}) = \mathbf{\Omega}_{l,m+3}(\mathrm{x}) = \mathbf{\Omega}_{m+3,l}(\mathrm{x}) = 0,\\
&\mathbf{\Omega}_{77}(\mathrm{x}) = \frac{\alpha}{\alpha_6}\lambda_2(\mathrm{x}), \quad \mathbf{\Omega}_{78}(\mathrm{x}) = \mathbf{\Omega}_{87}(\mathrm{x}) = \frac{-b_1}{\alpha_6}\lambda_2(\mathrm{x}),\\
&\mathbf{\Omega}_{88}(\mathrm{x}) = \frac{\gamma_0}{\alpha_6}\lambda_2(\mathrm{x}), \quad \mathbf{\Omega}_{99}(\mathrm{x}) = \frac{1}{k_3}\lambda_2(\mathrm{x}),\\
&\mathbf{\Omega}_{l+3,j+3}(\mathrm{x}) = \left(\frac{1}{\alpha_5}\mathrm{grad\,div} - \frac{1}{\gamma}\mathrm{curl\,curl}\right)\lambda_1(\mathrm{x}),\\
&\mathbf{\Omega}_{79}(\mathrm{x}) = \mathbf{\Omega}_{97}(\mathrm{x}) = 0, \quad \mathbf{\Omega}_{98}(\mathrm{x}) = \mathbf{\Omega}_{89}(\mathrm{x}) = 0,\\
&\lambda_1(\mathrm{x}) = -\frac{|\mathrm{x}|}{8\pi}, \quad \lambda_2(\mathrm{x}) = -\frac{1}{4\pi|\mathrm{x}|},\\
&l, j = 1,2,3 \qquad m = 4,5,6
\end{aligned}$$

Lemma 1.

If conditions (4) are satisfied, then

$$\Delta r_{l1}(\Delta) = g_1\Gamma_2(\Delta)H^*_{l1}(\Delta) - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1}$$
$$\Delta r_{l2}(\Delta) = g_1\Gamma_2(\Delta)H^*_{l2}(\Delta) - g_2[(\mu+\kappa)\Delta+\rho\omega^2)]\Gamma_1(\Delta)\delta_{l2}$$
$$l = 1,2,3,4,5 \qquad (33)$$

Proof: Using the equality

$$(\alpha_4\Delta+\rho\omega^2)H^*_{l1} - \Delta(c_0 H^*_{l3} + d_0 H^*_{l4} - \beta H^*_{l5}) = \frac{1}{g_1}\delta_{l1}\Gamma_1(\Delta) \qquad l = 1,2,3,4,5$$

Eq. $(22)_1$ implies that

$$\begin{aligned}\Delta r_{l1}(\Delta) &= g_1 g_2[(\gamma\Delta+\mu_1)\{(\alpha_4\Delta+\rho\omega^2)H^*_{l1} - \frac{1}{g_1}\delta_{l1}\Gamma_1(\Delta)\}\\ &\quad -\{(\lambda+\mu)(\gamma\Delta+\mu_1)-\kappa^2\}\Delta H^*_{l1}]\\ &= g_1 g_2[(\gamma\Delta+\mu_1)\{(\mu+\kappa)\Delta+\rho\omega^2\}+\kappa^2\Delta]H^*_{l1}\\ &\quad - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1}\\ &= g_1\Gamma_2(\Delta)H^*_{l1}(\Delta) - g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\delta_{l1}\end{aligned}$$

Similarly, from Eqs. $(22)_2$ and

$$(\alpha_5\Delta+\mu_1)H^*_{l2} + c_0\Delta H^*_{l3} + d_0\Delta H^*_{l4} = \frac{1}{g_1}\delta_{l2}\Gamma_1(\Delta) \qquad l = 1,2,3,4,5$$

we obtain

$$\begin{aligned}\Delta r_{l2}(\Delta) &= g_1 g_2\left[\left\{(\mu+\kappa)\Delta+\rho\omega^2)\right\}\left\{-c_0\Delta H^*_{l3} - d_0\Delta H^*_{l4} + (\alpha_5-\gamma)\Delta H^*_{l2}\right\} + \kappa^2\Delta H^*_{l2}\right]\\ &= g_1 g_2\left[\left\{(\gamma\Delta+\mu_1)\mathrm{H}^*_{l2} - \frac{1}{g_1}\delta_{l2}\Gamma_1(\Delta)\right\}\left\{(\mu+\kappa)\Delta+\rho\omega^2\right\} + \kappa^2\Delta H^*_{l2}\right]\\ &= g_1\Gamma_2(\Delta)H^*_{l2}(\Delta) - g_2[(\mu+\kappa)\Delta+\rho\omega^2)]\Gamma_1(\Delta)\delta_{l2}\end{aligned}$$

Lemma 2. If conditions (4) are satisfied and $\mathbf{x}\in\mathbf{R}^3\setminus\{\mathbf{0}\}$, then

$$\left[r_{l1}(-\xi_m^2)-\frac{g_2}{\xi_m^2}(-\gamma\xi_m^2+\mu_1)\Gamma_1(-\xi_m^2)\delta_{l1}\right]\eta_j(\mathrm{x})=-\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)\,\mathrm{H}_{l1}^*(-\xi_m^2)\eta_j(\mathrm{x})$$

$$\left[r_{l2}(-\xi_m^2)-\frac{g_2}{\xi_m^2}\{-(\mu+\kappa)\xi_m^2+\rho\omega^2\}\Gamma_1(-\xi_m^2)\delta_{l2}\right]\eta_j(\mathrm{x})=-\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)\,\mathrm{H}_{l2}^*(-\xi_m^2)\eta_j(\mathrm{x}) \qquad (34)$$

$$l=1,2,3,4,5$$

Proof. We obtain Eqs.(34) on using the following equality

$$\Delta\eta_j(\mathrm{x})=-\xi_m^2\eta_j(\mathrm{x})$$

in the system of Eqs.(32)

Theorem 2. If conditions (4) are satisfied and $\mathbf{x}\in\mathbf{R}^3\setminus\{\mathbf{0}\}$, then

$$\mathbf{\Theta}^{(1)}(\mathrm{x})=\operatorname{grad}\operatorname{div}\sum_{m=1}^{5}v_{1m}\eta_m(x)-\operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}v_{1e}\eta_e(\mathrm{x})$$

$$\mathbf{\Theta}^{(2)}(\mathrm{x})=\mathbf{\Theta}^{(3)}(\mathrm{x})=v_{20}\operatorname{curl}[\eta_6(\mathrm{x})-\eta_7(\mathrm{x})]+\operatorname{grad}\operatorname{div}\sum_{m=1}^{5}v_{2m}\eta_m(\mathrm{x})$$

$$\mathbf{\Theta}^{(4)}(\mathrm{x})=\operatorname{grad}\operatorname{div}\sum_{m=1}^{5}v_{1m}\eta_m(\mathrm{x})-\operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}v_{4e}\eta_m(\mathrm{x})$$

$$\mathbf{\Theta}_{er}^{(n)}(\mathrm{x})=\frac{\partial}{\partial x_e}\sum_{m=1}^{5}v_{nrj}\eta_m(\mathrm{x})$$

$$\mathbf{\Theta}_{re}^{(n+2)}(\mathrm{x})=\frac{\partial}{\partial x_e}\sum_{m=1}^{5}v_{n+2,rj}\eta_m(\mathrm{x})$$

$$\mathbf{\Theta}_{re}^{(9)}(\mathrm{x})=\sum_{m=1}^{5}v_{9qrj}\eta_m(\mathrm{x}) \quad e=1,2,3 \quad r,q=1,2 \quad n=5,6$$

where

$$\mathbf{\Theta}=\left\|\mathbf{\Theta}_{ej}\right\|_{8\times8}=\left\|\begin{matrix}\mathbf{\Theta}^{(1)} & \mathbf{\Theta}^{(2)} & \mathbf{\Theta}^{(5)}\\ \mathbf{\Theta}^{(3)} & \mathbf{\Theta}^{(4)} & \mathbf{\Theta}^{(6)}\\ \mathbf{\Theta}^{(7)} & \mathbf{\Theta}^{(8)} & \mathbf{\Theta}^{(9)}\end{matrix}\right\|_{9\times9} \qquad \mathbf{\Theta}^{(n)}=\left\|\mathbf{\Theta}_{ej}^{(n)}\right\|_{3\times3}$$

$$\mathbf{\Theta}^{(q)}=\left\|\mathbf{\Theta}_{ej}^{(q)}\right\|_{3\times2} \quad \mathbf{\Theta}^{(\mathrm{r})}=\left\|\mathbf{\Theta}_{ej}^{(\mathrm{r})}\right\|_{2\times3} \quad \mathbf{\Theta}^{(9)}=\left\|\mathbf{\Theta}_{ej}^{(9)}\right\|_{2\times2}$$

$$n=1,2,3,4 \quad q=5,6 \quad r=7,8$$

and

$$v_{1m}=-\frac{g_1}{\xi_m^2}s_{2j}H_{l1}^*(-\xi_m^2) \quad v_{1e}=\frac{(-1)^e g_2}{\xi_e^2(\xi_6^2-\xi_7^2)}(\gamma\xi_e^2-\mu_1)$$

$$v_{2m}=c_0 g_1(bk_3\xi_m^2-b\alpha_3-d\alpha_4+\beta\alpha_5) \qquad v_{20}=\frac{\kappa g_2}{\xi_6^2-\xi_7^2}$$

$$v_{4m}=-\frac{g_1}{\xi_m^2}s_{2m}H_{22}^*(-\xi_m^2) \quad v_{4e}=\frac{(-1)^e g_2}{k_e^2(k_6^2-k_7^2)}[(\mu+\kappa)\xi_m^2+\rho\omega^2]$$

$$v_{erm}=g_1 s_{2m}H_{e-4,r+2}^*(-\xi_m^2)$$

$$v_{e+2,rm}=g_1 s_{2m}H_{r+2,e-4}^*(-\xi_m^2)$$

$$v_{9qrm}=g_1 s_{2m}H_{q+2,r+2}^*(-\xi_m^2)$$

$$q,r=1,2 \quad m=1,2,3,4,5 \quad e=6,7 \tag{35}$$

Proof. On using

$$\mathbf{I}_{\eta_m}(\mathbf{x})=-\frac{1}{\xi_m^2}(\text{grad div -curl curl})\eta_m(\mathbf{x}) \quad \mathbf{x}\neq\mathbf{0}$$

and Eqs.(15),(26),(28),(30), we get

$$\begin{aligned}\Theta^{(1)}(\text{x})&=[g_2(\gamma\Delta+\mu_1)\Gamma_1(\Delta)\mathbf{I}+r_{11}(\Delta)\,\text{grad div}]\sum_{m=1}^{7}s_{1m}\eta_m(\text{x})\\&=\sum_{m=1}^{7}s_{1m}[\{r_{l1}(-\xi_m^2)-\frac{g_2}{\xi_m^2}(-\gamma\xi_m^2+\mu_1)\Gamma_1(-\xi_m^2)\}\,\text{grad div}\\&\qquad+\frac{g_2}{\xi_m^2}(-\gamma\xi_m^2+\mu_1)\Gamma_1(-\xi_m^2)\,\text{curl curl}]\eta_m(\text{x})\end{aligned} \tag{36}$$

Using $(34)_1$ in (36), we get

$$\Theta^{(1)}(\text{x})=\sum_{m=1}^{7}s_{1m}\left[\left\{-\frac{g_1}{\xi_m^2}\Gamma_2(-\xi_m^2)\,\text{H}_{l1}^*(-\xi_m^2)\right\}\text{grad div}+\frac{g_2}{\xi_m^2}(-\gamma\xi_m^2+\mu_1)\Gamma_1(-\xi_m^2)\,\text{curl curl}\right]\eta_m(\text{x})$$

By virtue of Eq. (35) and the equalities

$$\Gamma_1(-\xi_m^2)s_{1m}=\begin{cases}0, & m=1,2,3,4,5\\(-1)^m(\xi_6^2-\xi_7^2)^{-1} & m=6,7\end{cases}$$

$$\Gamma_2(-\xi_m^2)s_{1m} = \begin{cases} s_{2m}, & m = 1,2,3,4,5 \\ 0, & m = 6,7 \end{cases}$$

(37)

From Eq.(36), we obtain

$$\begin{aligned}\mathbf{\Theta}^{(1)}(\mathrm{x}) &= \operatorname{grad}\operatorname{div}\sum_{m=1}^{5}\left[-\frac{g_1}{\xi_m^2}s_{2m}H_{l1}^{*}(-\xi_m^2)\right]\eta_m(\mathrm{x}) \\ &\quad -\frac{g_2}{\xi_m^2}\operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}\frac{(-1)^e g_2(\gamma\xi_e^2-\mu_1)}{\xi_m^2(\xi_6^2-\xi_7^2)}\eta_e(\mathrm{x}) \\ &= \operatorname{grad}\operatorname{div}\sum_{m=1}^{5}v_{1m}\eta_m(\mathrm{x}) - \operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}v_{1e}\eta_e(\mathrm{x})\end{aligned}$$

Other formulae of Theorem 2 can be proved in the similar manner.

Theorem 3. The relations

$$\begin{aligned}&\mathbf{\Lambda}_{gh}(\mathrm{x}) - \mathbf{\Omega}_{gh}(\mathrm{x}) = \mathrm{const} + \mathrm{O}(|\mathrm{x}|) \\ &\frac{\partial^p}{\partial x_1^{p_1}\partial x_2^{p_2}\partial x_3^{p_3}}[\mathbf{\Lambda}_{gh}(\mathrm{x}) - \mathbf{\Omega}_{gh}(\mathrm{x})] = \mathrm{O}(|\mathrm{x}|^{1-p})\end{aligned} \tag{38}$$

and

$$|\Lambda_{ej}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1} \qquad |\Lambda_{e+3,j+3}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1} \qquad |\Lambda_{nn}(\mathbf{x})| < \mathrm{const}|\mathbf{x}|^{-1} \tag{39}$$

hold in the neighborhood of the origin, where
$p = p_1 + p_2 + p_3,\quad p \ge 1,\quad p_j \ge 0,\quad e, j = 1,2,3,\quad g,h = 1,2,...,9,\quad n = 7,8,9.$

Proof. It is evident from Theorem 2 and Corollary 2,

$$\mathbf{\Lambda}^{(1)}(\mathbf{x}) - \mathbf{\Omega}^{(1)}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) \tag{40}$$

where

$$\mathbf{G}(\mathbf{x}) = \|G_{em}(\mathrm{x})\|_{3\times 3} = \operatorname{grad}\operatorname{div}\eta^{(1)}(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\eta^{(2)}(\mathbf{x})$$

$$\begin{aligned}\eta^{(1)}(\mathbf{x}) &= \sum_{m=1}^{5}v_{1m}\eta_m(\mathbf{x}) - \frac{1}{\alpha_4}\lambda_1(\mathbf{x}) \\ \eta^{(2)}(\mathbf{x}) &= \sum_{e=6}^{7}v_{1e}\eta_e(\mathbf{x}) - \frac{1}{\mu+\kappa}\lambda_1(\mathbf{x})\end{aligned} \tag{41}$$

From Eq. (41), in the neighborhood of the origin , we have

$$\eta^{(1)}(\mathbf{x}) = -\frac{1}{8\pi}\left[2\sum_{m=1}^{5} v_{1m}\sum_{n=0}^{\infty}\frac{i^n\xi_m^n}{n!}|\mathbf{x}|^{n-1} - \frac{1}{\alpha_4}|\mathbf{x}|\right]$$
$$= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{m=1}^{5} v_{1m} - |\mathbf{x}|\left(\sum_{m=1}^{5} v_{1m}\xi_m^2 + \frac{1}{\alpha_4}\right)\right] - \frac{i}{4\pi}\sum_{m=1}^{5} v_{1m}\xi_m + \eta^{(3)}(\mathbf{x})$$
$$\eta^{(2)}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{e=6}^{7} v_{1e} - |\mathbf{x}|\left(\sum_{e=6}^{7} v_{1e}\xi_e^2 + \frac{1}{\mu+\kappa}\right)\right] - \frac{i}{4\pi}\sum_{e=6}^{7} v_{1e}\xi_e + \eta^{(4)}(\mathbf{x}) \tag{42}$$

where

$$\eta^{(3)}(\mathrm{x}) = -\frac{1}{4\pi}\sum_{m=1}^{5} v_{1m}\sum_{n=3}^{\infty}\frac{i^n\xi_m^n}{n!}|\mathrm{x}|^{n-1}$$
$$\eta^{(4)}(\mathrm{x}) = -\frac{1}{4\pi}\sum_{e=6}^{7} v_{1e}\sum_{n=3}^{\infty}\frac{i^n\xi_e^n}{n!}|\mathrm{x}|^{n-1} \tag{43}$$

Therefore, from Eq. (43), in the neighborhood of the origin , we obtain

$$\eta^{(p)}(\mathbf{x}) = \mathrm{O}(|\mathbf{x}|^2) \qquad \frac{\partial}{\partial x_e}\eta^{(p)}(\mathbf{x}) = \mathrm{O}(|\mathbf{x}|)$$
$$\frac{\partial^2}{\partial x_e \partial x_m}\eta^{(p)}(\mathbf{x}) = \mathrm{const} + \mathrm{O}(|\mathbf{x}|) \quad e,m = 1,2,3 \quad p = 3,4 \tag{44}$$

By virtue of Eq. (42) and the inequalities

$$\sum_{m=1}^{5} v_{1m} = \sum_{e=6}^{7} v_{1e} = -\frac{1}{\rho\omega^2}$$
$$\sum_{m=1}^{5} v_{1m}\xi_m^2 + \frac{1}{\alpha_4} = 0$$
$$\sum_{e=6}^{7} v_{1e}\xi_e^2 + \frac{1}{\mu+\kappa} = 0$$
$$(\mathrm{grad\,div} - \mathrm{curl\,curl})\frac{1}{|\mathbf{x}|} = \Delta\frac{1}{|\mathbf{x}|} = 0 \quad \mathbf{x} \neq \mathbf{0} \tag{45}$$

From Eq. (41), we get

$$\mathbf{G}(\mathbf{x}) = \mathrm{grad\,div}\,\eta^{(3)}(\mathrm{x}) - \mathrm{curl\,curl}\,\eta^{(4)}(\mathrm{x}) \tag{46}$$

On using Eqs.(44) and (45) in Eq. (40), we obtain the relation $(38)_1$ for $g,h=1,2,3$.

Similarly, other formulae of Eq. (38) can be proved.

We can obtain inequalities (39) from Eqs.(38) as

$$\left|\boldsymbol{\Omega}_{mn}(\mathrm{x})\right| < \mathrm{const}\left|\mathbf{x}\right|^{-1} \qquad \left|\boldsymbol{\Omega}_{m+3,n+3}(\mathbf{x})\right| < \mathrm{const}\left|\mathbf{x}\right|^{-1}$$
$$\left|\boldsymbol{\Omega}_{hh}(\mathrm{x})\right| < \mathrm{const}\left|\mathbf{x}\right|^{-1} \qquad m,n=1,2,3 \quad h=7,8,9$$

Hence, the matrix $\boldsymbol{\Omega}(\mathbf{x})$ is the singular part of the fundamental matrix $\boldsymbol{\Lambda}(\mathbf{x})$ in the neighborhood of the origin.

5. Special cases

(i) Neglecting the thermal and micropolarity effect in system of equations(3), yield the system of steady vibrations for homogeneous isotropic elastic material with double porosity as :

$$\begin{aligned}
&\left[\mu\Delta+\rho\omega^2\right]\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+b\,\mathrm{grad}\,\varphi+d\,\mathrm{grad}\,\psi=0\\
&(\alpha\Delta+\mu_2)\varphi+(b_1\Delta-\alpha_3)\psi-b\,\mathrm{div}\,\mathbf{u}=0\\
&(b_1\Delta-\alpha_3)\varphi+(\gamma_0\Delta+\mu_3)\psi-d\,\mathrm{div}\,\mathbf{u}=0
\end{aligned} \tag{47}$$

The derived fundamental solution for the system of equations (47) is similar as obtained by Svanadze [51] .

(ii) In the absence of single porosity parameter in the system of equations (3), we obtain the system of steady vibrations for homogeneous isotropic micropolar thermoelastic material with voids as

$$\begin{aligned}
&\left[(\mu+\kappa)\Delta+\rho\omega^2\right]\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\kappa\,\mathrm{curl}\boldsymbol{\Phi}+b\,\mathrm{grad}\,\varphi-\beta\,\mathrm{grad}\,T=0\\
&(\gamma\Delta+\mu_1)\boldsymbol{\Phi}+(\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\Phi}+\kappa\,\mathrm{curl}\mathbf{u}+c_0\,\mathrm{grad}\,\varphi=0\\
&(\alpha\Delta+\mu_2)\varphi-b\,\mathrm{div}\,\mathbf{u}+\gamma_1 T-c_0\,\mathrm{div}\,\boldsymbol{\Phi}=0\\
&(k_3\Delta-\rho C^*)T-\beta T_0\,\mathrm{div}\,\mathbf{u}+\gamma_1 T_0\varphi=0
\end{aligned} \tag{48}$$

Obtaining the fundamental solution of the system of equations (48) is same as given by Ciarlette et al.[45].

(iii) In absence of single porosity parameter and thermal effect in the system of equations(3), the system of steady vibrations for homogeneous isotropic micropolar elastic material with voids is

$$\left[(\mu+\kappa)\Delta+\rho\omega^2\right]\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\kappa\,\mathrm{curl}\boldsymbol{\Phi}+b\,\mathrm{grad}\varphi=0$$
$$(\gamma\Delta+\mu_1)\boldsymbol{\Phi}+(\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\Phi}+\kappa\,\mathrm{curl}\mathbf{u}+c_0\mathrm{grad}\varphi=0$$
$$(\alpha\Delta+\mu_2)\varphi-b\,\mathrm{div}\,\mathbf{u}-c_0\mathrm{div}\,\boldsymbol{\Phi}=0 \qquad (49)$$

The resulting fundamental solution obtained from the system of equations (49) is in agreement with those obtained by Scarpetta [42].

(iv) In absence of micropolarity effect, we obtain the system of steady vibrations for homogeneous isotropic thermoelastic material with double porosity as :

$$\begin{aligned}
&(\mu\Delta+\rho\omega^2)\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+b\,\mathrm{grad}\varphi+d\,\mathrm{grad}\psi-\beta\,\mathrm{grad}T=0\\
&(\alpha\Delta+\mu_2)\varphi+(b_1\Delta-\alpha_3)\psi-b\,\mathrm{div}\,\mathbf{u}+\gamma_1T=0\\
&(b_1\Delta-\alpha_3)\varphi+(\gamma_0\Delta+\mu_3)\psi-d\,\mathrm{div}\,\mathbf{u}+\gamma_2T=0\\
&(k_3\Delta-\rho C^*)T-\beta T_0\mathrm{div}\,\mathbf{u}-\gamma_1T_0\varphi-\gamma_2T_0\psi=0
\end{aligned} \qquad (50)$$

The derived fundamental solution from the system of equations (50) is similar as obtained by Scarpetta et al.[25] with some modification.

6. Concluding remarks

1. The constructed fundamental solution $\boldsymbol{\Lambda}(\mathbf{x})$ of the system (3) can be used

(i) to solve the boundary value problems by using boundary element method.

(ii) for constructing the surface and volume potentials and establishing their basic properties[40]

(iii) for investigating three-dimensional boundary value problems in micropolar thermoelastic materials with double porosity by potential method [40]

2. By the method applied in this paper, it is possible to represent the fundamental solutions of the systems of equations in the different theories of continuum mechanics.

REFERENCES

1. R. DE BOER, *Theory of Porous Media* , Springer-Verleg, New York, 2000.
2. R. DE BOER., W. EHLERS., *A Historical review of the foundation of porous media theories*, Acta Mech.,**74**, 1-8, 1988.
3. M. A. BIOT, *General theory of three-dimensional consolidation*, J. Appl. Phys., **12**,155-164, 1941.
4. R. M. BOWEN, *Incompressible Porous media models by use of the theory of mixtures*, Int. J. Engg. Sci, **18**,1129-1148, 1980.
5. R. DE BOER, W. EHLERS, *Uplift, friction and capillarity-Three fundamental effects for liquid saturated porous solids*, Int. J. Solid Struc.,**26,** 43-57, 1990.

6. G. I. BARENBLATT., I. P. ZHELTOV, I. N. KOCHINA, *Basic Concept in the theory of seepage of homogeneous liquids in fissured rocks (strata)*, J. Appl. Math. Mech.**24**, 1286-1303, 1960.
7. R. K. WILSON, E. C. AIFANTIS, *On the theory of consolidation with double porosity*, Int. J. Engg. Sci.,**20**(9), 1009-1035, 1982.
8. M. Y. KHALED, D. E. BESKOS, E. C. AIFANTIS, *On the theory of consolidation with double porosity-III,* Int.J. Numer.Analy. Meth. Geomech., **8**,101-123, 1984.
9. R. K. WILSON., E. C. AIFANTIS, *A Double Porosity Model for Acoustic Wave propagation in fractured porous rock,* Int. J. Engg. Sci., **22**, no. 8-10:1209-1227, 1984
10. D. E. BESKOS , E. C. AIFANTIS, *On the theory of consolidation with Double Porosity-II,* Int. J. Engg. Sci. , **24**(111), 1697-1716, 1986.
11. N. KHALILI, S. VALLIAPPAN, *Unified theory of flow and deformation in double porous media,* Eur.J. Mech. A, Solids, **15**, 321-336 , 1996.
12. E. C. AIFANTIS, *Introducing a multi –porous medium*, Developments in Mechanics, **8**, 209-211, 1977.
13. E. C. AIFANTIS, *On the response of fissured rocks*, Developments in Mechanics, **10**, 249-253, 1979.
14. E. C. AIFANTIS, On the Problem of Diffusion in Solids, Acta Mechanica.,**37**, 265-296, 1980.
15. E. C. AIFANTIS, *The mechanics of diffusion in solids,* T.A.M. Report No. 440, Dept. of Theor. Appl. Mech., University of Illinois, Urbana, Illinois, 1980.
16. K. N. MOUTSOPOULOS, I. E. ELEFTHERIADIS, E. C. AIFANTIS, *Numerical Simulation of Transport phenomena by using the double porosity/ diffusivity Continuum model*, Mechanics Research Communications, **23**(6), 577-582, 1996.
17. N. KHALILI , A. P. S. SELVADURAI, *A Fully Coupled Constitutive Model for Thermo-hydro –mechanical Analysis in Elastic Media with Double Porosity*, Geophys. Res. Lett., **30**, 2268-2271, 2003.
18. S. R .PRIDE., J. G.BERRYMAN., *Linear Dynamics of Double –Porosity Dual-Permeability Materials-I*, Phys. Rev. **E 68**,036603, 2003.
19. B. STRAUGHAN , *Stability and Uniqueness in Double Porosity Elasticity*, Int. J. Eng. Sci.,**65,** 1-8, 2013
20. M. SVANADZE , *Fundamental solution in the theory of consolidation with double porosity*, J.Mech. Behav. Mater.,**16,** 123-130, 2005.
21. M. SVANADZE, *Dynamical Problems on the Theory of Elasticity for Solids with Double Porosity*, Proc. Appl. Math. Mech.,**10,** 209-310, 2010
22. M. SVANADZE, *Plane Waves and Boundary Value Problems in the Theory of Elasticity for solids with Double Porosity*, Acta Appl. Math., **122,** 461-470, 2012.
23. M. SVANADZE , *On the Theory of Viscoelasticity for materials with Double Porosity*, Disc. and Cont. Dynam. Syst.Ser. B , **19(7),** 2335-2352, 2014.
24. M. SVANADZE , Uniqueness theorems in the theory of thermoelasticity for solids with double porosity, *Meccanica*, **49,** 2099-2108, 2014.

25. E. SCARPETTA, M. SVANADZE., V. ZAMPOLI, *Fundamental Solutions in the Theory of Thermoelasticity for Solids with Double Porosity,* J.Therm. Stresses, **37(6)**,727-748, 2014.
26. E. SCARPETTA , M. SVANADZE , *Uniqueness Theorems in the quasi-static Theory of Thermo elasticity for solids with Double Porosity*, J. Elasticity.,**117**, 2014.
27. J.W. NUNZIATO, S.C. COWIN, *A nonlinear theory of elastic materials with voids*, Arch. Rat. Mech. Anal., **72**, 175-201, 1979.
28. S.C. COWIN, J. W. NUNZIATO, *Linear elastic materials with voids,* J. Elasticity, **13**, 125-147, 1983.
29. D. IESAN., R. QUINTANILLA, *On a theory of thermoelastic materials with a double porosity structure,* J. Therm. Stresses, **37**, 1017-1036, 2014.
30. A. C. ERINGEN, *Linear theory of micropolar elasticity*, Journal of Applied Mathematics and Mechanics, **15**, 909-923, 1966.
31. A.C. ERINGEN , *Foundations of micropolar thermoelasticity*, International centre for Mechanical science, Udline Course and lectures, No. 23, Springer-Verlag, Berlin,1970.
32. W. NOWACKI, *Theory of Asymmetric Elasticity,* Oxford:Pergamon. 1986.
33. T. R. TOUCHERT, W.D. JR. CLAUS , T. ARIMAN. *The linear theory of micropolar thermoelasticity*, International journal of engineering science, **6**, 37-47, 1968.
34. D. S. CHANDRASEKHARAIAH, *Heat flux dependent micropolar thermoelasticity*, International journal of engineering science, **24**, 1389-1395,1986.
35. E. BOSCHI , D. IESAN, *A generalized theory of micropolar thermoelasticity*, Meccanica, **7**, 154-157, 1973.
36. R. B. HETNARSKI, *The fundamental solution of the coupled thermoelastic problemfor small times,* Archwm. Mech.Stosow., **16**, 23-31, 1964
37. R. B. HETNARSKI, *Solution of the coupled problem of thermoelasticity in form of a series of functions*, Archwm. Mech.Stosow., **16**, 919-941, 1964
38. L. HöRMANDER, *Linear Partial Differential operators.* Springer-Verlag: Berlin ,1963.
39. L. HöRMANDER, *The analysis of linear partial differential Operators II: Differential operators with constant coefficients.* Springer-Verlag: Berlin ,1983.
40. V. D. KUPRADZE, T. G. GEGELIA, M. O. BASHELEISHVILI, T. V. BURCHULADZE, *Three- Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, New York, Oxford, 1979.
41. R. BURRIDGE , C. A. VARGAS, *The Fundamental Solution in Dynamic Poroelasticity,* Geophysical Journal of the Royal Astronomical Society, **58**, 61–90, 1979.
42. E. SCARPETTA, *On the fundamental solution in micropolar elasticity with voids,* Acta Mechanica, **82,** 151–158, 1990.
43. R. DE BOER, M. SVANADZE, *Fundamental Solution of the System of Equations of Steady Oscillations in the Theory of Fluid-saturated Porous Media*, Transport in Porous Media, **56**, 39–50, 2004.
44. M. SVANADZE, *Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures*, J. Thermal stresses, **27**, 151-170, 2004.

45. M CIARLETTE., A. SCALIA, M. SVANADZE, *Fundamental solution in the theory of micropolar thermoelasticity for materials with voids*, J. Thermal stresses, **30**, 213-229, 2007.
46. M. SVANADZE , *Fundamental solutions of the system of equations of steady oscillations in the theory of thermomicrostretch elastic solids*, International journal of engineering science,**43**, 417-431,2005.
47. M. SVANADZE, R. TRACINA, *Representations of Solutions in the Theory of Thermoelasticity with Microtemperatures for Microstretch Solids*, Journal of Thermal Stresses, **34**, 161–178, 2011.
48. R. KUMAR, T.KANSAL, *Fundamental solution in the theory of thermomicrostretch elastic diffusive solids*, ISRN Applied Mathematics, 1-15, 2011.
49. R. KUMAR, T.KANSAL, *Fundamental solution in the theory of micropolar thermoelastic diffusion with voids*, computational and applied mathematics, **31**, 169-189, 2012.
50. K. SHARMA, P. KUMAR, *Propagation of plane waves and fundamental solution in thermoviscoelastic medium with voids,* Journal of thermal stresses, **36**, 94-111, 2013.
51. M. SVANADZE., *Fundamental solutions in the linear theory of consolidation for elastic solids with double porosity*, Journal of mathematical science, **195**, no. 2, 258-268, 2013 .
52. S. SHARMA, K. SHARMA, R. R. BHARGAV, *Plane wave and fundamental solution in an electro-microstretch elastic solids*, Afrika Matematika, **25**, 483-497, 2014.
53. R. KUMAR, D. TANEJA, K. KUMAR, *Fundamental and plane wave solution in swelling pororus medium,* Journal of the African mathematical Union, Afrika matematika, **25**, no.2, 397-410, 2014.
54. R. KUMAR, M. KAUR , S. C. RAJVANSHI, *Representation of fundamental and plane waves solutions in the theory of micropolar generalized thermoelastic solid with two temperatures*, Journal of Computational and Theoretical Nanoscience, **12**, 691-702, 2015.
55. T. GEGELIA , L. JENTSCH, *Potential Methods in Continuum Mechanics*, Georgian mathematical Journal, **1**, 599-640, 1994.
56. M. MARIN, S. VLASE, M. PAUN, *Considerations on double porosity structure for micropolar bodies*, AIP Advances, **5**, doi:10.1063/1.4914912, 2015.