Formulas for the H/V ratio of Rayleigh waves in incompressible pre-stressed half-spaces

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Abstract

In this paper, the propagation of a Rayleigh wave in an incompressible prestressed elastic half-space is considered. The main aim is to derive exact formulas for the H/V ratio, the ratio of the amplitude of the horizontal displacement to the amplitude of the vertical displacement of the Rayleigh wave. First, the H/V ratio equations are obtained using the secular equation and the relation between the H/V ratio and the Rayleigh wave velocity. Then, the exact formulas for the H/V ratio have been derived for a general strain-energy function by analytically solving the H/V ratio equations. These formulas are then specified to several particular strain-energy functions. Since the obtained formulas are exact and totally explicit, they will be a good tool for nondestructively evaluating pre-stresses of structures before and during loading.

Key words: Rayleigh waves, Incompressible, Pre-stressed, The H/V ratio, Formula for the H/V ratio.

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1 Introduction

Pre-stressed materials are widely used in many technical applications recently. That makes the nondestructive evaluation of pre-stresses in structures before and during loading become necessary and important. For this task, the usage of Rayleigh wave is a good choice, see for example [1]-[5]. A Rayleigh wave is excited first and it propagates in the structures whose pre-stresses are needed to identify. Then, its velocity is measured. An inverse problem is formulated to determine the pre-stresses based on that measured velocity and on the explicit secular equations.

One of the advantages of using H/V ratio to the velocity in the inverse problem is that H/V ratio is more sensitive to the state of stress than the Rayleigh velocity as shown recently by M. Junge et al. [6]. Further, in contrast to the Rayleigh velocity, H/V ratio is reference-free and dimensionless. Therefore the H/V ratio of Rayleigh waves is a more convenient tool than the Rayleigh wave velocity for characterizing the state of stress. In the technique of using the H/V ratio, the explicit H/V ratio equations are considered as the mathematical base for extracting the pre-stresses from measured data of the H/V ratio. The inverse problem will become much more simple if the H/V ratio formulas are given in explicit form.

The main aim of this paper is to derive the explicit exact H/V ratio formulas for incompressible pre-stressed elastic half-spaces. This is done by establishing the explicit H/V ratio equations first by using the secular equation and the relation between the H/V ratio and the Rayleigh wave velocity. This relation is obtained by using the surface impedance matrix of Rayleigh waves propagating in incompressible pre-stressed half-spaces. Then the H/V ratio equations are solved analytically to derive the explicit exact H/V ratio formulas for a general strain-energy function. These formulas are then specified to several particular strain-energy functions. The obtained formulas express directly and explicitly the H/V ratio in terms of material parameters and pre-stresses. Since the obtained H/V ratio formulas are totally explicit, they will be a powerful tool in the evaluation of pre-stresses appearing in structures before and during loading.

It is worth to note that, the H/V ratio is an important quantity which reflects fundamental properties of the elastic material [7], and is used as a well-known method in seismology [8, 9]. It can be used for the nondestructive evaluation of the elastic constants of material [10] as well.

2 Surface impedance matrix of Rayleigh waves in incompressible pre-stressed half-spaces

2.1. Surface impedance matrix for elastic half-spaces

Consider a Rayleigh wave propagating on the surface of an elastic half-space occupying the region $x_2 \geq 0$ with the velocity $c\,(> 0)$, the wave number $k\,(> 0)$ in the x_1-direction and decaying in the x_2-direction. Then, the displacement vector $\mathbf{u}$ and the traction vector $\mathbf{t}$ at the planes $x_2 =$ const of the Rayleigh wave are of the form:

$$\mathbf{u} = \mathbf{U}(y)\mathrm{e}^{ik(x_1-ct)},\ \mathbf{t} = ik\mathbf{\Sigma}(y)\mathrm{e}^{ik(x_1-ct)},\ y = k\,x_2 \tag{2.1}$$

Matrix $\mathbf{M}$ is called the surface impedance matrix of the Rayleigh wave if it relates $\mathbf{U}(0)$ and $\mathbf{\Sigma}(0)$ by the equality [11]:

$$\mathbf{\Sigma}(0) = i\mathbf{M}\mathbf{U}(0) \tag{2.2}$$

It is well-known that matrix $\mathbf{M}$ is an important tool for studying the existence and uniqueness of Rayleigh waves in generally anisotropic solids [11].

According to (2.2), for a Rayleigh wave propagating in a traction-free elastic half-space, its secular equation is: $|\mathbf{M}| = 0$. Therefore we can obtain immediately explicit secular equations of Rayleigh waves if the corresponding surface impedance matrices is expressed in explicit form.

2.2. Surface impedance matrix for incompressible pre-stressed half-spaces

Consider the initial state of an unstressed body of incompressible isotropic elastic material occupying the half-space $X_2 \geq 0$ and then it is assumed to be deformed to a new configuration by being applied a pure homogeneous strain of the form:

$$x_1 = \lambda_1 X_1,\ x_2 = \lambda_2 X_2,\ x_3 = \lambda_3 X_3,\ \lambda_i = \text{const},\ \ i = 1, 2, 3 \tag{2.3}$$

where $\lambda_i > 0$ are the principal stretches of the deformation. The deformed configuration the body, therefore, occupies the region $x_2 \geq 0$.

In the deformed configuration we consider a Rayleigh wave propagating with the velocity $c\,(> 0)$, the wave number $k\,(> 0)$ in the x_1-direction and decaying in the x_2-direction. According to Dowaikh & Ogden [12] and Vinh [13], the Rayleigh wave is a two-component surface wave: $\mathbf{U} = [U_1\ U_2]^T$, $\mathbf{\Sigma} = [\Sigma_1\ \Sigma_2]^T$ with:

$$U_1 = -(b_1 B_1 e^{-b_1 y} + b_2 B_2 e^{-b_2 y}),\ U_2 = -i(B_1 e^{-b_1 y} + B_2 e^{-b_2 y}),$$
$$\Sigma_1 = -i(\beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}),\ \Sigma_2 = -(\eta_1 B_1 e^{-b_1 y} + \eta_2 B_2 e^{-b_2 y}) \qquad (2.4)$$

where: B_1, B_2 are constants, b_1, b_2 are two roots with positive real part of the equation:

$$b^4 - Sb^2 + P = 0 \qquad (2.5)$$

in which S and P are given by:

$$S = \frac{2\beta - X}{\gamma},\ P = \frac{\alpha - X}{\gamma} \qquad (2.6)$$

and:

$$X = \rho c^2,\ \beta_k = \gamma b_k^2 + \gamma_*,\ \eta_k = \big[X - (2\beta + \gamma_*) + \gamma b_k^2\big] b_k,\ k = 1, 2 \qquad (2.7)$$

The quantities α, β, γ and γ_* are defined as:

$$\alpha = B_{1212},\ \gamma = B_{2121},\ 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221},\ \gamma_* = \gamma - \sigma_2 \qquad (2.8)$$

where σ_2 is the principal Cauchy pre-stress along the x_2-direction [12, 13], B_{ijkl} are components of the fourth order elasticity tensor defined as follows [12, 13, 14]:

$$B_{iijj} = \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}$$
$$B_{ijij} = \begin{cases} (\lambda_i \frac{\partial W}{\partial \lambda_i} - \lambda_j \frac{\partial W}{\partial \lambda_j}) \frac{\lambda_i^2}{\lambda_i^2 - \lambda_j^2}, & (i \neq j, \lambda_i \neq \lambda_j) \\ \frac{1}{2}(B_{iiii} - B_{iijj} + \lambda_i \frac{\partial W}{\partial \lambda_i}) & (i \neq j, \lambda_i = \lambda_j) \end{cases} \qquad (2.9)$$
$$B_{ijji} = B_{jiij} = B_{ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i}\ (i \neq j)$$

for $i, j \in \{1, 2, 3\}, W = W(\lambda_1, \lambda_2, \lambda_3)$ (noting that $\lambda_1\lambda_2\lambda_3 = 1$) is the strain-energy function per unit volume, all other components being zero, the summation convention does not apply in (2.9). In the stress-free configuration (2.9) reduces to:

$$B_{iiii} = B_{ijij} = \mu \ (i \neq j), B_{iijj} = B_{ijji} = 0 \ (i \neq j) \tag{2.10}$$

From the strong-ellipticity condition, it follows that [12, 14]:

$$\alpha > 0, \gamma > 0 \tag{2.11}$$

It has been shown that [13, 15], if a Rayleigh wave exists, then:

$$0 < X < \alpha \tag{2.12}$$

and:

$$P > 0, \ S + 2\sqrt{P} > 0 \tag{2.13}$$

Taking $x_2 = 0$ in (2.4) we have:

$$\begin{aligned} U_1(0) &= -(b_1 B_1 + b_2 B_2), \ U_2(0) = -i(B_1 + B_2), \\ \Sigma_1(0) &= -i(\beta_1 B_1 + \beta_2 B_2), \ \Sigma_2(0) = -(\eta_1 B_1 + \eta_2 B_2) \end{aligned} \tag{2.14}$$

After eliminating B_1, B_2 from (2.14) we arrive at the impedance matrix for an incompressible pre-stressed half-space (see also Vinh et al. [16], pages 182,183), namely:

$$\mathbf{M} = \begin{bmatrix} [\beta] & -i[b;\beta] \\ -i[\eta] & -[b;\eta] \end{bmatrix} \tag{2.15}$$

Here we use the notations:

$$[f; g] := f_2 g_1 - f_1 g_2, \ [f] := f_2 - f_1 \tag{2.16}$$

Using (2.7) it is not difficult to verify that:

$$[\beta] = \gamma\sqrt{S + 2\sqrt{P}}, \ [b; \beta] = (\gamma_* - \gamma\sqrt{P}), \ [b; \eta] = -\gamma\sqrt{P}\sqrt{S + 2\sqrt{P}}, \ [\eta] = -[b; \beta] \tag{2.17}$$

Thus **M** is of the form:

$$\mathbf{M} = \begin{bmatrix} M_{11} & iM_{12} \\ -iM_{12} & M_{22} \end{bmatrix} \tag{2.18}$$

where M_{ik} are real and given by:

$$M_{11} = \gamma\sqrt{S + 2\sqrt{P}}, \ M_{12} = \gamma\sqrt{P} - \gamma_*, \ M_{22} = \gamma\sqrt{P}\sqrt{S + 2\sqrt{P}} \tag{2.19}$$

It is clear from (2.18), (2.19) and (2.13) that **M** is hermitian.

Remark 1: It is clear from (2.13), (2.18) and (2.19) that M_{ik} are real and:

$$M_{11} > 0, \ M_{22} > 0 \tag{2.20}$$

3 Equations for the H/V ratio

• *Secular equation:*

Consider a Rayleigh wave propagating in an incompressible deformed isotropic elastic half-space $x_2 \geq 0$, as described in Subsection 2.2, with the velocity $c\,(> 0)$, the wave number $k\,(> 0)$, in the x_1-direction and decaying in the x_2-direction. Let **M** is the surface impedance matrix of the Rayleigh wave. Then it is given by (2.18), (2.19). Suppose that the half-space is free of traction, i.e $\mathbf{\Sigma}(0) = \mathbf{0}$. Then, the secular equation of the Rayleigh wave is: $\det\mathbf{M} = 0$. Using the expressions of M_{ij} given by (2.19) in this equation gives the secular equation [13]:

$$\gamma(\alpha - X) + (2\beta + 2\gamma_* - X)\sqrt{\gamma(\alpha - X)} - \gamma_*^2 = 0 \tag{3.1}$$

In terms of the dimensionless parameters:

$$\delta_1 = \gamma/\alpha\,(> 0),\ \ \delta_2 = \beta/\alpha,\ \ \delta_3 = \gamma_*/\alpha \tag{3.2}$$

the secular equation (3.1) becomes:

$$\delta_1(1 - x) + \sqrt{\delta_1}\big(2\delta_2 + 2\delta_3 - x\big)\sqrt{1 - x} - \delta_3^2 = 0,\ \ 0 < x < 1 \tag{3.3}$$

where $x = c^2/c_2^2$, $c_2 = \sqrt{\alpha/\rho}$, $0 < x < 1$ by (2.12).

- *Relation between the H/V ratio and the Rayleigh wave velocity:*

From the equation $\mathbf{MU}(0) = \mathbf{0}$ and (2.18):

$$\begin{cases} M_{11}\dfrac{U_1(0)}{U_2(0)} + iM_{12} = 0 \\ -iM_{12}\dfrac{U_1(0)}{U_2(0)} + M_{22} = 0 \end{cases} \tag{3.4}$$

From (3.4) it follows:

$$\Big[\frac{U_1(0)}{U_2(0)}\Big]^2 = -\frac{M_{22}}{M_{11}} \tag{3.5}$$

Since: $-M_{22}/M_{11} < 0$ by Remark 1, Eq. (3.5) provides:

$$\frac{U_1(0)}{U_2(0)} = i\sqrt{\frac{M_{22}}{M_{11}}} \tag{3.6}$$

By the definition of H/V ratio $\kappa = |U_1(0)/U_2(0)|$, thus we have:

$$\kappa = \sqrt{\frac{M_{22}}{M_{11}}}\ \Rightarrow \kappa^2 = \frac{M_{22}}{M_{11}} \tag{3.7}$$

Introducing the expressions of M_{11} and M_{22} given by (2.19) into the second of (3.7) we arrive at:

$$\kappa^2 = \frac{\sqrt{1-x}}{\sqrt{\delta_1}} \tag{3.8}$$

This is the desired relation between the H/V ratio and the Rayleigh wave velocity.

Remark 2: Since $0 < x < 1$, it follows from (3.8): $0 < \kappa^2 < 1$ if $\alpha < \gamma$; κ^2 may go to ∞ if α is much more large than γ.

- *Equations for the H/ V ratio:*

Putting $w = \kappa^2$, from (3.8) wa have:

$$w = \frac{\sqrt{1-x}}{\sqrt{\delta_1}}, \ 0 < x < 1 \tag{3.9}$$

Eliminating x from (3.3) and (3.9) yields a cubic equation for w (provided $\delta_3 \neq 0$):

$$f_1(w) := w^3 + w^2 + a_1 w + a_0 = 0, \ w \in \big(0, \ \delta_1^{-1/2}\big) \tag{3.10}$$

where:

$$a_0 = -\frac{\delta_3^2}{\delta_1^2}, \ a_1 = (2\delta_2 + 2\delta_3 - 1)/\delta_1 \tag{3.11}$$

Equation (3.10) is the equation determining the H/V ratio. *It is interesting that Eq. (3.10) for κ^2 is just Eq. (5.26) in Ref. [12] for η.*

If $\delta_3 = 0 \Rightarrow a_0 = 0$, then Eq. (3.10) is equivalent to a quadratic equation, namely:

$$f_2(w) := w^2 + w + a_1 = 0, \ w \in \big(0, \ \delta_1^{-1/2}\big) \tag{3.12}$$

By w_r we denote a root of Eq. (3.10) or (3.12) that belong to the interval $(0, \ \delta^{-1/2})$.

• *Existence of solution of the H/V ratio equations:*

Proposition 1. Suppose $\delta_3 \neq 0$, then Eq. (3.10) has a unique root w_r if and only if:

$$\sqrt{\delta_1} + 2\delta_2 + 2\delta_3 - \frac{\delta_3^2}{\sqrt{\delta_1}} > 0 \tag{3.13}$$

Proof: Let $\delta_3 \neq 0$. From (3.10) we have:

$$f_1(\delta_1^{-1/2}) = \frac{1}{\delta_1\sqrt{\delta_1}}\Big(\sqrt{\delta_1} + 2\delta_2 + 2\delta_3 - \frac{\delta_3^2}{\sqrt{\delta_1}}\Big) \tag{3.14}$$

Let $\Delta' = 1 - 3a_1$ be the discriminant of equation $f_1'(w) = 3w^2 + 2w + a_1 = 0$.

If $\Delta' \leq 0$: $\Rightarrow f_1'(w) \geq 0$, $\forall w \in (-\infty, +\infty) \Rightarrow f_1(w)$ is strictly monotonically increasing in $(-\infty, +\infty)$ so in $(0, +\infty)$. As $f_1(0) = -\delta_3^2/\sqrt{\delta_1} < 0$ and $f_1(+\infty) = +\infty$, equation $f_1(w) = 0$ has exact one real root in the interval $(0, +\infty)$. Using (3.14) it is easily to show that this root is w_r if (3.13) holds and it is not w_r if (3.13) is not satisfied.

If $\Delta' > 0$: $\Rightarrow$ equation $f_1'(w) = 0$ has two distinct roots, denoted by w_{max} and w_{min} so that either $w_{max} < w_{min} \leq 0$ or $w_{max} < 0 < w_{min}$ due to $w_{max} + w_{min} = -2/3 < 0$.

If $w_{max} < w_{min} \leq 0$: because $f_1(0) < 0$, $f_1(+\infty) = +\infty$ and $f_1(w)$ is strictly monotonically increasing in $(0, +\infty)$, equation $f_1(w) = 0$ has therefore a unique root in $(0, +\infty)$. Due to (3.14), this root is w_r if (3.13) holds and it is not w_r if (3.13) is not valid.

If $w_{max} < 0 < w_{min}$: since $f_1(w)$ is strictly monotonically decreasing in (w_{max}, w_{min}) and $f_1(0) < 0$ it follows: $f_1(w_{min}) < 0$. As $f_1(w)$ is strictly monotonically increasing in $(w_{min}, +\infty)$, $f_1(w_{min}) < 0$ and $f_1(+\infty) = +\infty$, equation $f_1(w) = 0$ has therefore a unique root in $(w_{min}, +\infty)$ so in $(0, +\infty)$. From (3.14) it follows: this root is w_r if (3.13) is valid and it is not w_r if (3.13) does not hold. The proof of Proposition 1 is completed.

The proof of Proposition 1 for the case $\Delta' > 0$ shows immediately that:

Proposition 2. Suppose $\delta_3 \neq 0$ and (3.13) holds. If Eq. (3.10) has two or three distinct real roots, then w_r is the largest root.

Proposition 3. Let $\delta_3 = 0$, then Eq. (3.12) has a unique root w_r if and only if:

$$-\sqrt{\delta_1} < 2\delta_2 < 1 \tag{3.15}$$

Proof: It follows from (3.12) that:

$$f_2(\delta_1^{-1/2}) = \frac{1}{\delta_1}(\sqrt{\delta_1} + 2\delta_2) \tag{3.16}$$

(i) Let $\delta_3 = 0$ and (3.15) holds. From $(3.15)_2$ we have: $a_1 = (2\delta_2 - 1)/\delta_1 < 0 \Rightarrow$ equation $f_2(w) = 0$ has two distinct roots w_1 and w_2 so that: $w_1 < 0 < w_2$ because $w_1.w_2 = a_1 < 0$. This says that Eq. (3.12) has a unique root w_2 in $(0, +\infty)$. From (3.16) and $(3.15)_1$ it follows: $f_2(\delta_1^{-1/2}) > 0$. Since $f_2(0) = a_1 < 0 \Rightarrow w_2 \in (0, \delta^{-1/2}) \Rightarrow w_2 = w_r$.

(ii) If $2\delta_2 \geq 1 \Rightarrow$ either Eq. (3.12) has no roots or it has two negative roots w_1, w_2 because $w_1 + w_2 = -1$ and $w_1.w_2 = a_1 \geq 0$. That means Eq. (3.12) has no root w_r.

(iii) If $2\delta_2 \leq -\sqrt{\delta_1}$ and $2\delta_2 > 1$. According to the argument of (i), Eq. (3.12) has a unique root $w_2 \in (0, +\infty)$, but it is not w_r because $2\delta_2 \leq -\sqrt{\delta_1}$ and (3.16).

(iv) If $2\delta_2 \leq -\sqrt{\delta_1}$ and $2\delta_2 \geq 1$. Following the argument of (ii) yields that Eq. (3.12) has no root w_r. The proof is finished.

Remark 3: Since the mapping (3.9) is a 1-1 mapping that maps $x \in (0,\, 1)$ to $w \in \big(0,\, \delta_1^{-1/2}\big)$, the conditions (3.13) and (3.15) are also necessary and sufficient for a Rayleigh wave to exist.

4 Formulas for the H/V ratio for a general strain-energy function

Theorem 1

If there exists a Rayleigh wave propagating along the x_1-direction, and attenuating in the x_2-direction, in an incompressible isotropic elastic half-space subject to a homogeneous initial deformation (Eq. (2.3)), then it is unique, and its squared H/V ratio κ^2 is determined as follows:

i) If $\delta_3 \neq 0$:

$$\kappa^2 = -\frac{1}{3} + \sqrt[3]{R + \sqrt{D}} + \frac{(1 - 3a_1)}{9\sqrt[3]{R + \sqrt{D}}} \tag{4.1}$$

where each radical is understood as the complex root taking its principal value, R and D are given by:

$$R = (9a_1 - 27a_0 - 2)/54,\;\; D = (4a_0 - a_1^2 - 18a_0 a_1 + 27a_0^2 + 4a_1^3)/108 \tag{4.2}$$

a_0 and a_1 are determined by (3.11).

ii) If $\delta_3 = 0$:

$$\kappa^2 = \frac{\sqrt{\delta_1 - 8\delta_2 + 4} - \sqrt{\delta_1}}{2\sqrt{\delta_1}} \tag{4.3}$$

Proof: The uniqueness of Rayleigh waves follows immediately from Propositions 1, 3. Now we present the derivation of the formulas (4.1) and (4.3).

(i) Suppose $\delta_3 \neq 0$ and (3.13) holds. Then, a unique Rayleigh wave can propagate in the half-space, according to Proposition 1, and its H/V ratio is determined by Eq. (3.10). Let $z = w + 1/3$, then in terms of z Eq. (3.10) has the form:

$$z^3 - 3q^2 z + r = 0 \tag{4.4}$$

where:

$$r = -2R,\ q^2 = \frac{(a_2^2 - 3a_1)}{9} \tag{4.5}$$

According to the theory of cubic equation, three roots $z_k\,(k = 1, 2, 3)$ of Eq. (4.4) are calculated by [17]:

$$z_1 = S + T,\ z_2 = -\frac{1}{2}(S+T) + \frac{i\sqrt{3}}{2}(S-T),\ z_3 = -\frac{1}{2}(S+T) - \frac{i\sqrt{3}}{2}(S-T) \tag{4.6}$$

where:

$$S = \sqrt[3]{R + \sqrt{D}},\ T = \sqrt[3]{R - \sqrt{D}},\ D = R^2 + Q^3,\ Q = -q^2 \tag{4.7}$$

In relation to the formulas (4.7) we emphasize two points:

+ The cube root of a negative real number is taken as the real negative root.

+ If, in the expression S, $R + \sqrt{D}$ is complex, the phase angle in T is taken as the negative of the phase angle in S so that $T = S^*$ where S^* is the complex conjugate of S.

Remark 4:

+ If $D > 0$, then Eq. (4.4) has one real root and two complex conjugate roots.

+ If $D = 0$, this equation has three real roots, at least two of which are equal.

+ If $D < 0$, it has three real distinct roots.

Let $z_r = 1/3 + w_r$, then z_r is a real root of Eq. (4.4) and if Eq. (4.4) has two or three real roots, z_r is the largest real root, according to Proposition 2. We will prove that z_r is given by:

$$z_r = \sqrt[3]{R + \sqrt{D}} + \frac{q^2}{\sqrt[3]{R + \sqrt{D}}} \tag{4.8}$$

where each radical is understood as a complex root taking its principal value, R and D are calculated by (4.2), q^2 is given by $(4.5)_2$. Formula (4.1) is obtained immediately from (4.8) and the relation $w_r = -1/3 + z_r$. We consider the distinct cases dependent on the values of D for proving (4.8).

• For the values of $D > 0$, according to Remark 4, Eq. (4.4) has a unique real root, so it is z_r, given by $(4.6)_1$:

$$z_r = \sqrt[3]{R + \sqrt{D}} + \sqrt[3]{R - \sqrt{D}} \tag{4.9}$$

in which the radicals are understood as real ones. To prove (4.8) we have to show that the right side of (4.9) in which the radicals being understood as real ones coincides with the right side of (4.8) where each radical being understood as a complex root taking its principal value. Since:

$$\sqrt[3]{R - \sqrt{D}} = \frac{q^2}{\sqrt[3]{R + \sqrt{D}}} \tag{4.10}$$

it is sufficient to prove that $R + \sqrt{D} > 0$. Note that, since Eq. (4.4) has a unique real root, so does Eq. (3.10). To prove $R + \sqrt{D} > 0$ we examine the distinct cases dependent on the values of Δ', the discriminant of equation $f_1'(w) = 0$.

- If $\Delta' \leq 0$, then $f_2(w)$ is strictly monotonically increasing in $(-\infty, +\infty)$. By w_N we denote the abscissa of the point of inflexion N of the cubic curve $y = f_1(w)$, then $w_N = -2/3 < 0$. This and the fact $f_1(0) = a_0 < 0$ and the strictly increasing monotonousness of $f_1(w)$ lead to $f_1(w_N) < 0$. Since $r = f_1(w_N)$ it follows $r < 0$, or equivalently $R > 0$. This leads to $R + \sqrt{D} > 0$.

- If $\Delta' > 0 \Rightarrow f_1'(w) = 0$ has two distinct roots w_{max} and w_{min} and either $w_{max} < w_{min} \leq 0$ or $w_{max} < 0 < w_{min}$ (see Proposition 1). In both two cases we always have: $f_1(w_{min}) < 0$. As Eq. (3.10) has a unique real root as addressed above it follows: $f_1(w_{max})f_1(w_{min}) > 0$, consequently, $f_1(w_{max}) < 0$. This and $f_1(w_{min}) < 0$ provides $r = f_1(w_N) < 0 \Rightarrow R = -r/2 > 0$, therefore we have $R + \sqrt{D} > 0$.

• For $D = 0$, analogously as above, one can see that $r < 0$, consequently $R > 0$. When $D = 0$ we have $R^2 = -Q^3 = q^6\,(q > 0) \Rightarrow R = q^3 \Rightarrow r = -2R = -2q^3$, so Eq. (4.4) becomes $z^3 - 3q^2z - 2q^3 = 0$ whose roots are: $z_1 = 2q$, $z_2 = -q$ (double root). This says $z_r = 2q$, since it is the largest root. With the help of $q > 0$ and $D = 0$ it is readily to see that z_r calculated by (4.8) is $2q$.

• For the values of $D < 0$, according to Remark 4, Eq. (4.4) has three distinct real roots and z_r is the largest one. Using the arguments presented in Ref. [18] (page 255), it is not difficult verify that in this case the largest real root of Eq. (4.4) is:

$$z_r = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}} \tag{4.11}$$

in which each radical is understood as a complex root taking its principal value. By 3θ we denote the phase angle of $R + i\sqrt{-D}$. Then, it is not difficult to prove that:

$$\sqrt[3]{R+\sqrt{D}} = q\mathrm{e}^{i\theta},\ \sqrt[3]{R-\sqrt{D}} = q\mathrm{e}^{-i\theta} \tag{4.12}$$

where radicals are understood as complex roots taking their principal value. From (4.12) we have immediately (4.10) and then (4.8) by taking into account (4.11).

(ii) Let $\delta_3 = 0$. According to Proposition 3, a Raleigh wave can propagate in the half-space if and only if (3.15) holds and the H/V ratio is computed by Eq. (3.12). According to the proof of Proposition 3, when (3.15) is valid, the quadratic equation (3.12): $f_2(w) = 0$ has two distinct real roots w_1 and w_2 so that: $w_1 < 0 < w_2$ and $w_2 = w_r$. It is readily to verify that w_2, so w_r, is calculated by formula (4.3). The proof of Theorem 1 is completed

5 Formulas of the H/V for specific strain-energy functions

5.1 The neo-Hookean material

For this material the strain-energy function is of the form [12]:

$$W(\lambda_1, \lambda_2) = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3) \tag{5.1}$$

where μ is a Lame coefficient. Consider the plain strain with $\lambda_3 = 1$. From (2.8), (2.9) and (5.1) we have:

$$\alpha = \mu\lambda^2,\ \gamma = \frac{\mu}{\lambda^2},\ \beta = \frac{\mu}{2}(\lambda^2 + \frac{1}{\lambda^2}),\ \gamma_* = \frac{\mu}{\lambda^2} - \sigma_2 \tag{5.2}$$

thus, according to (3.2) and (5.2):

$$\delta_1 = \frac{1}{\lambda^4},\ \delta_2 = \frac{1}{2}\big(1 + \frac{1}{\lambda^4}\big),\ \delta_3 = \frac{1 - \lambda^2\bar{\sigma}_2}{\lambda^4} \tag{5.3}$$

where $\lambda := \lambda_1$, $\bar{\sigma}_2 = \sigma_2/\mu$. Note that $\lambda_2 = \lambda^{-1}$ due to the incompressibility condition $\lambda_1\lambda_1\lambda_3 = 1$ and $\lambda_3 = 1$. From (3.11) and (5.3), the coefficients a_0 and a_1 of the H/V ratio equation (3.10) are:

$$a_1 = 3 - 2\lambda^2\bar{\sigma}_2,\ a_0 = -(1 - \lambda^2\bar{\sigma}_2)^2 \tag{5.4}$$

According to Proposition 1 and Theorem 1 (i), if $\delta_3 \neq 0$, i. e. $(1 - \lambda^2\bar{\sigma}_2) \neq 0$, then a Rayleigh wave is possible if only if (coming from (3.13) and (5.3)):

$$(1 + \lambda - \lambda^2 + \lambda^3 - \lambda^2\bar{\sigma}_2)(-1 + \lambda + \lambda^2 + \lambda^3 + \lambda^2\bar{\sigma}_2) > 0 \tag{5.5}$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by $(5.4)_1$, R and D are calculated by:

$$R = \frac{26}{27} + \frac{\lambda^2\bar{\sigma}_2(-8 + 3\lambda^2\bar{\sigma}_2)}{6},\ D = \frac{(-2 + \lambda^2\bar{\sigma}_2)^2(44 - 68\lambda^2\bar{\sigma}_2 + 27\lambda^4\bar{\sigma}_2^2)}{108} \tag{5.6}$$

From (5.3) it is readily to see that $2\delta_2 > 1$, the condition (3.15) is therefore not satisfied. According to Proposition 3, a Rayleigh wave is impossible for the case $\delta_3 = 0$, i. e.:

$$1 - \lambda^2\bar{\sigma}_2 = 0 \tag{5.7}$$

Thus, the H/V ratio is not defined at points in the space of λ and $\bar{\sigma}_2$ satisfying (5.7). When $\bar{\sigma}_2 = 0$, from (5.4) and (5.6) it follows:

$$a_1 = 3,\ R = \frac{26}{27},\ D = \frac{44}{27} \tag{5.8}$$

and using (4.1) and (5.17) gives: $w_r = 0.2956$. The condition (5.5) for this case is $\lambda > 0.5437$.

Figure 1 shows some contour lines of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ (shaded) given by (5.5). The thick continuous curve is the set of points at which the H/V ratio does not exist and it is defined by Eq. (5.7). The H/V ratio tends to zero when $(\lambda, \bar{\sigma}_2)$ approaching this curve. If λ is fixed and $\bar{\sigma}_2$ varies in its possible range, the squared H/V ratio approaches the supremum value $\delta_1^{-1/2} = \lambda^2$ at the two end points as stated in Remark 4.

Figure 2 shows the dependence of the H/V ratio on λ and $\bar{\sigma}_2$ computed by the exact formula (4.1) (along with (5.4), (5.6)). The left figure shows the dependence of the H/V ratio on λ with two fixed values of $\bar{\sigma}_2$ being 0 and 1. Each curve starts from a so-called cut-off value of λ computed by (5.5). When $\bar{\sigma}_2 = 0$, the H/V ratio is independent of λ and equals 0.2956 as mentioned above. The cut-off value of λ in this case is 0.5437. For $\bar{\sigma}_2 = 1$ the cut-off value of λ is 0.4656 and the squared H/V ratio starts from 0.2168 which is the square of 0.4656 according to Remark 4. For the case $\bar{\sigma}_2 = 1$ the exact curve first decreases and approaches zero when λ tending to $\lambda_0 = 1$ (the root of Eq. (5.7) with $\bar{\sigma}_2 = 1$) and then increase with λ.

The right figure shows the dependence of the H/V ratio on $\bar{\sigma}_2$ with two fixed

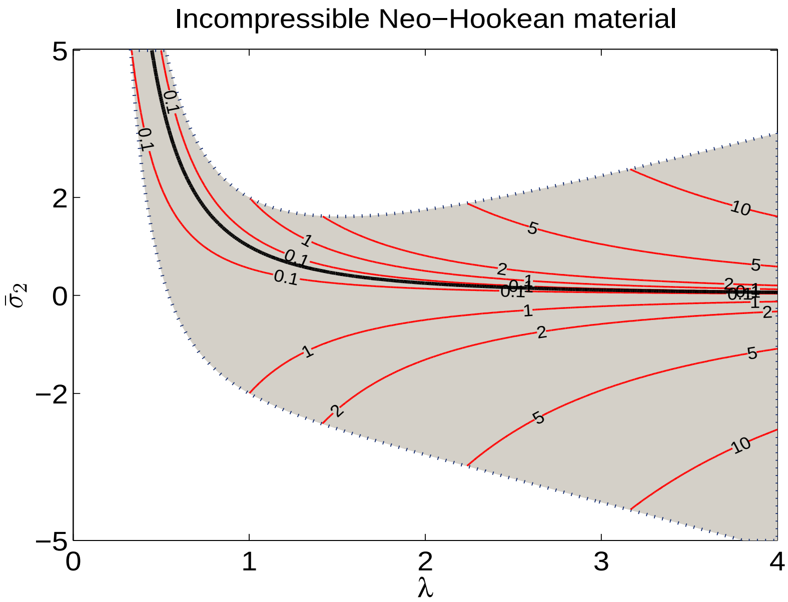

Figure 1: Some contours of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ (shaded) for the neo-Hookean material.

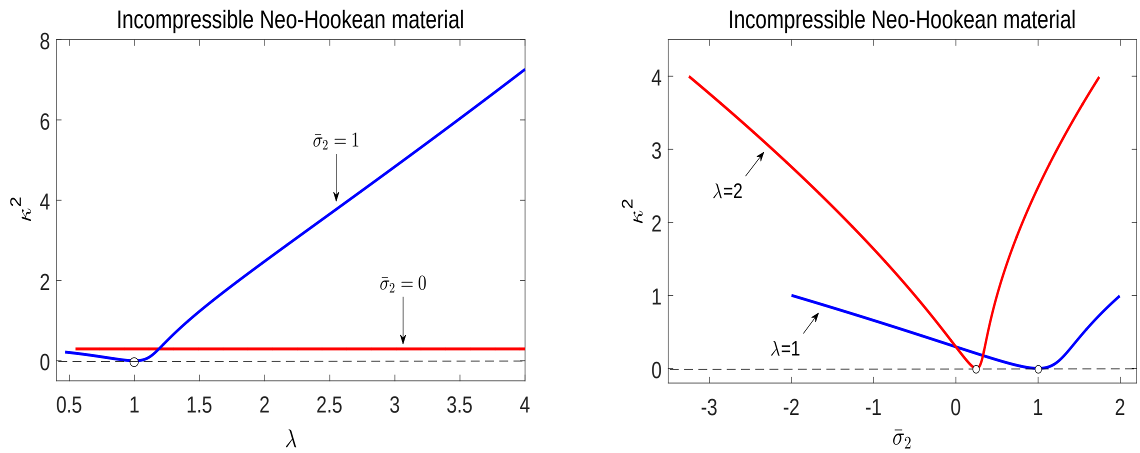

Figure 2: The H/V ratio curves computed by exact formula as a function of λ (left) and $\bar{\sigma}_2$ (right) with $\lambda_3 = 1$ for the neo-Hookean material.

values of λ: $\lambda = 1$ and $\lambda = 2$. The range of $\bar{\sigma}_2$ for the case $\lambda = 1$ is from -2 to 2 and it is from -3.25 to 1.75 for $\lambda = 2$. The H/V ratio curves take a "V" shape with two supremum points $\delta_1^{-1/2} = \lambda^2$ at the two end points, according to Remark

4, which are 1 and 4 for $\lambda = 1$ and 2, respectively. They have an infimum point: at $\bar{\sigma}_2 = 1$ for $\lambda = 1$ and at $\bar{\sigma}_2 = 0.25$ for $\lambda = 2$, following (5.7). The H/V ratio tends to zero when $\bar{\sigma}_2$ approaching 1 (for the case $\lambda = 1$) and 0.25 (for the case $\lambda = 2$).

5.2 The Varga material

For the Varga material, the strain-energy function takes the form [12]:

$$W(\lambda_1, \lambda_2) = 2\mu(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1\lambda_2} - 3) \tag{5.9}$$

Consider $\lambda_3 = 1$ and denote $\lambda_1 = \lambda$, from (2.8), (2.9) and (5.9) we have:

$$\alpha = 2\mu\frac{\lambda^3}{1+\lambda^2},\ \gamma = 2\mu\frac{1}{\lambda(1+\lambda^2)},\ \beta = 2\mu\frac{\lambda}{1+\lambda^2},\ \gamma_* = 2\mu\frac{1}{\lambda(1+\lambda^2)} - \sigma_2 \tag{5.10}$$

From (3.2) and (5.10) it follows:

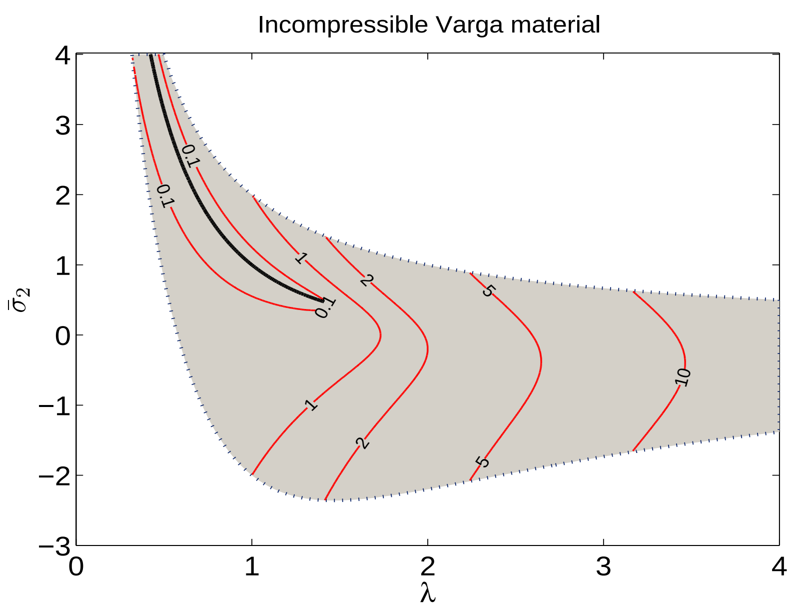

Figure 3: Some contours of the H/V-ratio in the possible space of λ and $\bar{\sigma}_2$ (shaded) for the Varga material with $\lambda_3 = 1$.

$$\delta_1 = \frac{1}{\lambda^4},\ \delta_2 = \frac{1}{\lambda^2},\ \delta_3 = \frac{2 - \lambda(1+\lambda^2)\bar{\sigma}_2}{2\lambda^4} \tag{5.11}$$

From (3.11) and (5.10), the coefficients a_0 and a_1 of the H/V ratio equation (3.10) are:

$$a_1 = -(\lambda^4 + \lambda^3\bar{\sigma}_2 - 2\lambda^2 + \lambda\bar{\sigma}_2 - 2),\ a_0 = -\left[\frac{\lambda(1+\lambda^2)\bar{\sigma}_2 - 2}{2}\right]^2 \tag{5.12}$$

According to Proposition 1 and Theorem 1 (i), if $\delta_3 \neq 0$, i. e. $[2 - \lambda(1+\lambda^2)\bar{\sigma}_2] \neq 0$, then a Rayleigh wave exists if only if (following from (3.13) and (5.10)):

$$(2 - \lambda\bar{\sigma}_2)(\lambda^3\bar{\sigma}_2 + 6\lambda^2 + \lambda\bar{\sigma}_2 - 2) > 0 \tag{5.13}$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by $(5.12)_1$, R and D are calculated by (4.2) and (5.12). If $\delta_3 = 0$, i. e.:

$$\lambda(1+\lambda^2)\bar{\sigma}_2 - 2 = 0 \tag{5.14}$$

then a Rayleigh wave can propagate in the half-space if only if (originating form (3.15) and (5.10)):

$$\lambda > \sqrt{2} \tag{5.15}$$

and from (4.3) the H/V ratio of the Rayleigh wave is given by:

$$w_r := \kappa^2 = -\frac{1}{2} + \frac{1}{2}\sqrt{4\lambda^4 - 8\lambda^2 + 1},\ \lambda > \sqrt{2} \tag{5.16}$$

When $\bar{\sigma}_2 = 0$, from (5.10), (5.12) and (4.2) we have:

$$\delta_3 = \frac{1}{\lambda^4} > 0,\ a_0 = -1,\ a_1 = 2 + 2\lambda^2 - \lambda^4 \tag{5.17}$$

therefore by Proposition 1 and Theorem 1 (i), a Rayleigh wave is possible if only if $\lambda > 1/\sqrt{3}$ (following from (5.13) with $\bar{\sigma}_2 = 0$) and the H/V ratio is given by (4.1) in which a_1 is given by $(5.17)_3$ and:

$$R = \frac{43}{54} + \frac{1}{3}\lambda^2 - \frac{1}{6}\lambda^4, \ D = -(\lambda^2-3)(\lambda^2+1)(4\lambda^8 - 16\lambda^6 + 5\lambda^4 + 22\lambda^2 + 29)/108 \quad (5.18)$$

Unlike the neo-Hookean material, the H/V ratio depends on λ for the case $\bar{\sigma}_2 = 0$.

Figure 3 shows some contour lines of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ defined by condition given in (5.13). When $(\lambda, \bar{\sigma}_2)$ approaching the boundary of this domain, the squared H/V ratio tends to λ^2, according to Remark 4. The thick continuous curve is expressed by Eq. (5.14) with $0 < \lambda \leq \sqrt{2}$. The H/V ratio goes to zero when $(\lambda, \bar{\sigma}_2)$ approaching this curve.

The left (right) figure in Fig. 4 shows the dependence of the H/V ratio on λ ($\bar{\sigma}_2$) with two fixed values of $\bar{\sigma}_2 = 0; 1$ (of $\lambda = 1; 2$) that is calculated by the exact formula (4.1) (along with (4.2). In the left figure 4, for $\bar{\sigma}_2 = 0$, the cut-off value of λ is 0.5773 determined by (5.13), for the case $\bar{\sigma}_2 = 1$, the range of λ is (0.4836, 2). In the right figure 4, the range of $\bar{\sigma}_2$ is $(-2, 2)$ for $\lambda = 1$ and $(-11/5, 1)$ for $\lambda = 2$ and at the end points, the squared H/V ratio approaches λ^2. On both figures, there is a point (marked by circles) with $\bar{\sigma}_2 = 1$ and $\lambda = 1$ at which the Rayleigh waves do not exist. This point belongs to the thick curve shown in Fig. 3.

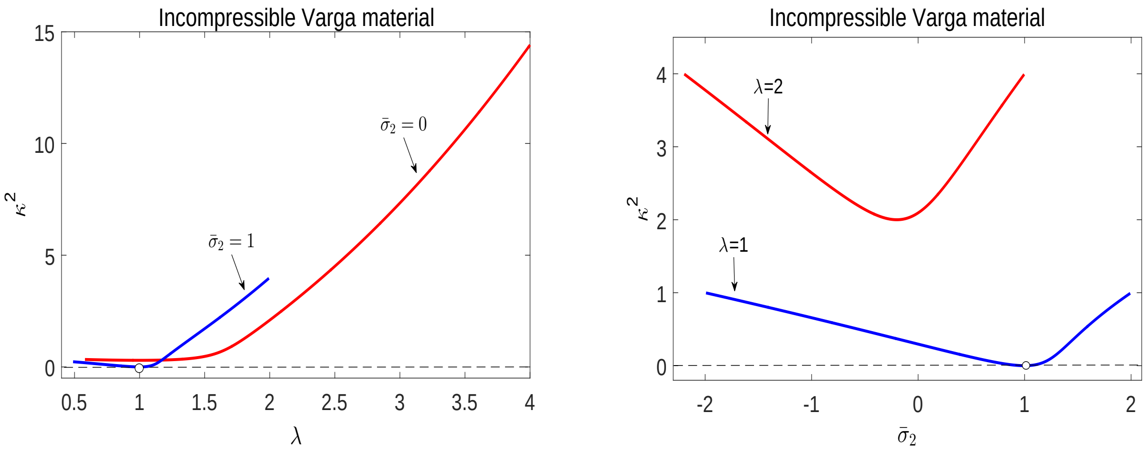

Figure 4: The squared H/V ratio computed by exact formula as a function of λ (left) and $\bar{\sigma}_2$ (right) with $\lambda_3 = 1$ for incompressible Varga material.

5.3 The m=1/2 material

For the $m = 1/2$ material, the strain-energy function is of the form [12]:

$$W(\lambda_1, \lambda_2) = 8\mu(\lambda_1^{1/2} + \lambda_2^{1/2} + \frac{1}{\lambda_1^{1/2}\lambda_2^{1/2}} - 3) \tag{5.19}$$

Consider $\lambda_3 = 1$ and denote $\lambda_1 = \lambda$, from (2.8), (2.9) and (5.19) we have:

$$\begin{aligned} \alpha &= \frac{4\mu\lambda^4}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, \; \gamma = \frac{4\mu}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, \\ \beta &= \frac{\mu(-\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda - 1)}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, \; \gamma_* = \gamma - \sigma_2 \end{aligned} \tag{5.20}$$

From (3.2) and (5.22) it follows:

$$\delta_1 = \frac{1}{\lambda^4}, \; \delta_2 = -\frac{1}{4} + \frac{1}{2\lambda} + \frac{1}{2\lambda^2} + \frac{1}{2\lambda^3} - \frac{1}{4\lambda^4}), \; \delta_3 = \frac{4 - \sqrt{\lambda}(\lambda+1)(\lambda^2+1)\bar{\sigma}_2}{4\lambda^4} \tag{5.21}$$

From (3.11) and (5.22), the coefficients of the H/V ratio equation (3.10) are:

$$\begin{aligned} a_1 &= (1+\lambda)(1+\lambda^2)(1 - \sqrt{\lambda}\bar{\sigma}_2/2) + \frac{1 - 3\lambda^4}{2}, \\ a_0 &= -\left(\frac{\sqrt{\lambda}(1+\lambda)(1+\lambda^2)\bar{\sigma}_2}{4} - 1\right)^2 \end{aligned} \tag{5.22}$$

According to Proposition 1 and Theorem 1 (i), if $\delta_3 \neq 0$, i. e. $[4 - \sqrt{\lambda}(\lambda+1)(\lambda^2+1)\bar{\sigma}_2] \neq 0$, then a Rayleigh wave exists if only if (following from (3.13) and (5.21)):

$$8(\lambda^2 - 4\lambda + 2) + 8\sqrt{\lambda}(\lambda - 1)\bar{\sigma}_2 + \lambda(\lambda^2+1)\bar{\sigma}_2^2 < 0 \tag{5.23}$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by $(5.22)_1$, R and D are calculated by (4.2) and (5.22).

In case $\bar{\sigma}_2 = 0$, the equation of H/V ratio becomes

$$w^3 + w^2 + (\frac{3}{2} + \lambda + \lambda^2 + \lambda^3 - \frac{3}{2}\lambda^4)w - 1 = 0 \tag{5.24}$$

and its solution depends on the principal stresses λ, unlike the case of Neo-Hookean's material.

Fig. 5 shows some contour lines of squared H/V ratio in the domain of λ and $\bar{\sigma}_2$ in which the Rayleigh surface waves exist. Unlike the Neo-Hookean and Varga materials, this domain is bounded in λ. The picture of contour lines in this material is similar to that of Varga material. The thick continuous curve shows the set of points at which H/V ratio is not defined. In this case, $\lambda < 1.3756$, and H/V ratio approaches to zero around this curve.

Fig. 6 shows the dependence of squared H/V ratio on λ (and $\bar{\sigma}_2$) using the exact, (4.1) and (5.6). In the left figure, for $\bar{\sigma}_2 = 0$, Rayleigh waves exist in $2 - \sqrt{2} < \lambda < 2 + \sqrt{2}$. For $\bar{\sigma}_2 = 1$, λ varies from 0.4896 to 1.7734. The range of $\bar{\sigma}_2$ on the right figure is $(-2, 2)$ and $(\frac{2}{5}(-\sqrt{2} - 2\sqrt{3}), \frac{2}{5}(-\sqrt{2} + 2\sqrt{3}))$ for λ equals 1 and 2 respectively.

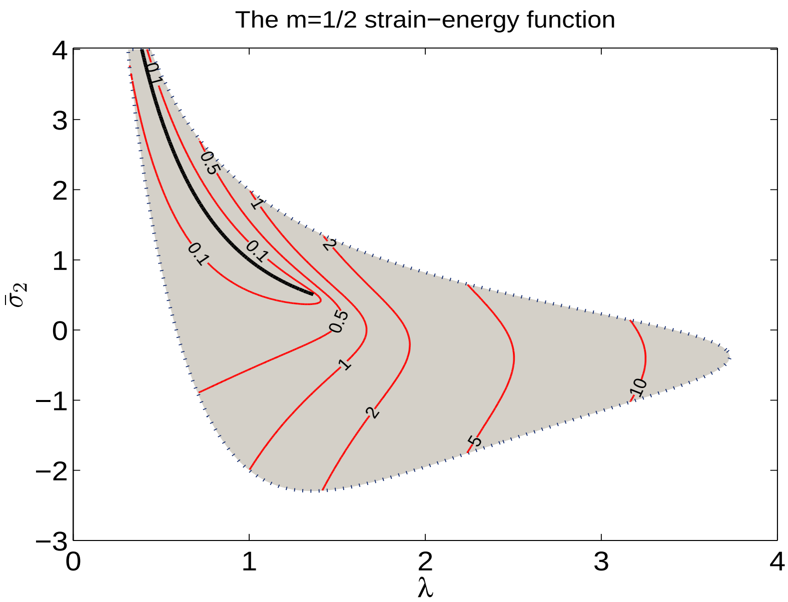

Figure 5: Some contours of different values of squared H/V-ratio in space of λ and $\bar{\sigma}_2$ for incompressible $m = 1/2$ material with $\lambda_3 = 1$.

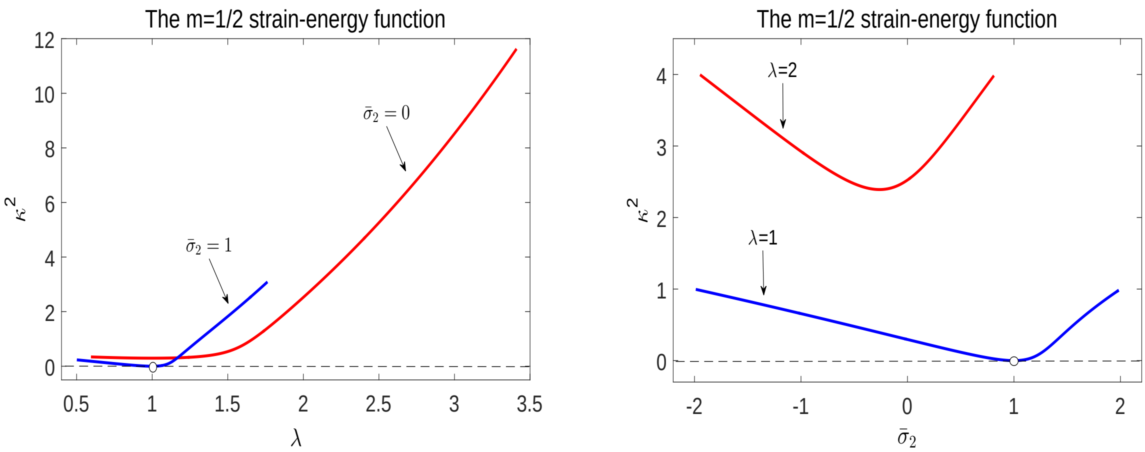

Figure 6: The squared H/V ratio computed by exact formula as a function of λ (left) and $\bar{\sigma}_2$ (right) with $\lambda_3 = 1$ for incompressible $m = 1/2$ material.

6 Conclusions

In this paper, the exact H/V ratio formulas have been derived by solving analytically the H/V ratio equations. These formulas are valid for a general strain-energy function. Several particular strain-energy functions are employed to specify these formulas. Some numerical examples are carried out to examine the dependance of the H/V ratio on the pre-stress. Since the H/V ratio is a convenient tool for nondestructively evaluating pre-stresses of structures before and during loading, the obtained formulas will be significant in practical applications.

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