

Formulas for the H/V ratio of Rayleigh waves in incompressible pre-stressed half-spaces

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IN THIS PAPER, THE PROPAGATION OF A RAYLEIGH WAVE in an incompressible pre-stressed elastic half-space is considered. The main aim is to derive exact formulas for the H/V ratio, the ratio of the amplitude of the horizontal displacement to the amplitude of the vertical displacement of the Rayleigh wave. First, the H/V ratio equations are obtained using the secular equation and the relation between the H/V ratio and the Rayleigh wave velocity. Then, the exact formulas for the H/V ratio have been derived for a general strain-energy function by analytically solving the H/V ratio equations. These formulas are then specified to several particular strain-energy functions. Since the obtained formulas are exact and totally explicit, they will be a good tool for nondestructively evaluating pre-stresses of structures before and during loading.

Key words: Rayleigh waves, incompressible, pre-stressed, the H/V ratio, formula for the H/V ratio.

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1. Introduction

SINCE PRE-STRESSED MATERIALS ARE WIDELY USED in many technical applications nowadays, the nondestructive evaluation of pre-stresses in structures before and during loading becomes necessary and important. For this task, the usage of the Rayleigh wave is a good choice, see for example [1]–[5]. A Rayleigh wave is excited first and it propagates in pre-stressed structures that need to be characterized. Then, its velocity is measured. An inverse problem is formulated to determine pre-stress based on that measured velocity and on the explicit secular equations [6]–[10] or the wave velocity formulas [11]–[14].

In comparison with the Rayleigh wave velocity, the Rayleigh wave H/V ratio is a more convenient tool for nondestructively evaluating pre-stress due to two reasons. Firstly, the H/V ratio is more easily measured than the wave velocity because it is independent of the distance between the exciting wave point and the receiving wave point and the time with which the wave propagates in this distance [15]. Secondly, while measured (*non-dimensional*) values of H/V ratio are directly used for solving the inverse problem, in order to use measured (*dimensional*) values of the Rayleigh wave velocity for solving the inverse problem we must evaluate the shear wave velocity because the secular equations and the formulas for Rayleigh waves include (squared) the dimensionless Rayleigh wave velocity that is the ratio between the Rayleigh wave velocity and the shear wave velocity. The non-dimensional nature of the H/V ratio makes it better suited for nondestructively evaluating pre-stress. In the technique of using the H/V ratio, the explicit H/V ratio equations are considered as the mathematical base for extracting the pre-stresses from measured data of the H/V ratio. The inverse problem will become much more simple if the H/V ratio formulas are given in an explicit form.

The main aim of this paper is to derive the explicit exact H/V ratio formulas for incompressible pre-stressed elastic half-spaces. This is done by establishing the explicit H/V ratio equations first by using the secular equation and the relation between the H/V ratio and the Rayleigh wave velocity. This relation is obtained by using the surface impedance matrix of a Rayleigh wave propagating in incompressible pre-stressed half-spaces. Then the H/V ratio equations are solved analytically to derive the explicit exact H/V ratio formulas for a general strain-energy function. These formulas are then specified to several specific strain-energy functions. The obtained formulas express directly and explicitly the H/V ratio in terms of material parameters and pre-stresses. Since the obtained H/V ratio formulas are totally explicit, they will be a powerful tool in the evaluation of pre-stresses appearing in structures before and during loading.

It is worth to note that, the H/V ratio is an important quantity which reflects fundamental properties of the elastic material [16]. Therefore, it can be used for the nondestructive evaluation of the elastic constants of material as well.

2. Surface impedance matrix of Rayleigh waves in incompressible pre-stressed half-spaces

2.1. Surface impedance matrix for elastic half-spaces

Consider a Rayleigh wave propagating on the surface of an elastic half-space occupying the region $x_2 \geq 0$ with the velocity $c(> 0)$, the wave number $k(> 0)$

in the x_1 -direction and decaying in the x_2 -direction. Then, the displacement vector \mathbf{u} and the traction vector \mathbf{t} at the planes $x_2 = \text{const}$ of the Rayleigh wave are of the form:

$$(2.1) \quad \mathbf{u} = \mathbf{U}(y)e^{ik(x_1-ct)}, \quad \mathbf{t} = ik\mathbf{\Sigma}(y)e^{ik(x_1-ct)}, \quad y = kx_2.$$

The matrix \mathbf{M} is called the surface impedance matrix of the Rayleigh wave if it relates $\mathbf{U}(0)$ and $\mathbf{\Sigma}(0)$ by the equality [17]:

$$(2.2) \quad \mathbf{\Sigma}(0) = i\mathbf{M}\mathbf{U}(0).$$

It is well-known that the matrix \mathbf{M} is an important tool for studying the existence and uniqueness of Rayleigh waves in generally anisotropic solids [17].

According to (2.2), for the Rayleigh wave propagating in a traction-free elastic half-space, its secular equation is: $|\mathbf{M}| = 0$. Therefore we can obtain immediately explicit secular equations of Rayleigh waves if the corresponding surface impedance matrices is expressed in an explicit form.

2.2. Surface impedance matrix for incompressible pre-stressed half-spaces

Consider the initial state of an unstressed body of incompressible isotropic elastic material occupying the half-space $X_2 \geq 0$ and then it is assumed to be deformed by the application of a pure homogeneous strain of the form

$$(2.3) \quad x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \quad \lambda_i = \text{const}, \quad i = 1, 2, 3,$$

where $\lambda_i > 0$ are the principal stretches of the deformation. In the deformed configuration the body, therefore, occupies the region $x_2 \geq 0$.

In the deformed configuration we consider a Rayleigh wave propagating with the velocity $c(> 0)$, the wave number $k(> 0)$ in the x_1 -direction and decaying in the x_2 -direction. According to DOWAIKH and OGDEN [6] and VINH [11], the Rayleigh wave is a two-component surface wave: $\mathbf{U} = [U_1 \ U_2]^T$, $\mathbf{\Sigma} = [\Sigma_1 \ \Sigma_2]^T$ with

$$(2.4) \quad \begin{aligned} U_1 &= -(b_1 B_1 e^{-b_1 y} + b_2 B_2 e^{-b_2 y}), & U_2 &= -i(B_1 e^{-b_1 y} + B_2 e^{-b_2 y}), \\ \Sigma_1 &= -i(\beta_1 B_1 e^{-b_1 y} + \beta_2 B_2 e^{-b_2 y}), & \Sigma_2 &= -(\eta_1 B_1 e^{-b_1 y} + \eta_2 B_2 e^{-b_2 y}), \end{aligned}$$

where: B_1, B_2 are constants, b_1, b_2 are two roots with positive real part of the equation

$$(2.5) \quad b^4 - Sb^2 + P = 0$$

in which S and P are given by

$$(2.6) \quad S = \frac{2\beta - X}{\gamma}, \quad P = \frac{\alpha - X}{\gamma}$$

and

$$(2.7) \quad X = \rho c^2, \quad \beta_k = \gamma b_k^2 + \gamma_*, \quad \eta_k = [X - (2\beta + \gamma_*) + \gamma b_k^2] b_k, \quad k = 1, 2.$$

The quantities α , β , γ and γ_* are defined as

$$(2.8) \quad \alpha = B_{1212}, \quad \gamma = B_{2121}, \quad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221}, \quad \gamma_* = \gamma - \sigma_2,$$

where σ_2 is the principal Cauchy pre-stress along the x_2 -direction [6, 11], B_{ijkl} are components of the fourth order elasticity tensor defined as follows [6, 11, 18]:

$$(2.9) \quad \begin{aligned} B_{iijj} &= \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}, \\ B_{ijij} &= \begin{cases} \left(\lambda_i \frac{\partial W}{\partial \lambda_i} - \lambda_j \frac{\partial W}{\partial \lambda_j} \right) \frac{\lambda_i^2}{\lambda_i^2 - \lambda_j^2}, & (i \neq j, \lambda_i \neq \lambda_j), \\ \frac{1}{2} \left(B_{iiii} - B_{iijj} + \lambda_i \frac{\partial W}{\partial \lambda_i} \right) & (i \neq j, \lambda_i = \lambda_j), \end{cases} \\ B_{ijji} &= B_{jii j} = B_{ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} \quad (i \neq j), \end{aligned}$$

for $i, j \in \{1, 2, 3\}$, $W = W(\lambda_1, \lambda_2, \lambda_3)$ (noting that $\lambda_1 \lambda_2 \lambda_3 = 1$) is the strain-energy function per unit volume, all other components being zero, the summation convention does not apply in (2.9). In the stress-free configuration (2.9) reduces to

$$(2.10) \quad B_{iiii} = B_{ijij} = \mu \quad (i \neq j), \quad B_{iijj} = B_{ijji} = 0 \quad (i \neq j).$$

From the strong-ellipticity condition, it follows that [6, 18]:

$$(2.11) \quad \alpha > 0, \quad \gamma > 0.$$

It has been shown that [11, 19], if a Rayleigh wave exists, then:

$$(2.12) \quad 0 < X < \alpha$$

and

$$(2.13) \quad P > 0, \quad S + 2\sqrt{P} > 0.$$

Taking $y = 0$ in (2.4) we have:

$$(2.14) \quad \begin{aligned} U_1(0) &= -(b_1 B_1 + b_2 B_2), & U_2(0) &= -i(B_1 + B_2), \\ \Sigma_1(0) &= -i(\beta_1 B_1 + \beta_2 B_2), & \Sigma_2(0) &= -(\eta_1 B_1 + \eta_2 B_2). \end{aligned}$$

After eliminating B_1, B_2 from (2.14) we arrive at the impedance matrix for an incompressible pre-stressed half-space (see also VINH *et al.* [20], pp. 182, 183), namely:

$$(2.15) \quad \mathbf{M} = \frac{1}{[b]} \begin{bmatrix} [\beta] & -i[b; \beta] \\ -i[\eta] & -[b; \eta] \end{bmatrix}.$$

Here we use the notations:

$$(2.16) \quad [f; g] := f_2 g_1 - f_1 g_2, \quad [f] := f_2 - f_1.$$

Using (2.7) it is not difficult to verify that:

$$(2.17) \quad \begin{aligned} \frac{[\beta]}{[b]} &= \gamma \sqrt{S + 2\sqrt{P}}, & \frac{[b; \beta]}{[b]} &= (\gamma_* - \gamma \sqrt{P}), \\ \frac{[b; \eta]}{[b]} &= -\gamma \sqrt{P} \sqrt{S + 2\sqrt{P}}, & [\eta] &= -[b; \beta]. \end{aligned}$$

Thus \mathbf{M} is of the form:

$$(2.18) \quad \mathbf{M} = \begin{bmatrix} M_{11} & iM_{12} \\ -iM_{12} & M_{22} \end{bmatrix},$$

where M_{ik} are real and given by:

$$(2.19) \quad M_{11} = \gamma \sqrt{S + 2\sqrt{P}}, \quad M_{12} = \gamma \sqrt{P} - \gamma_*, \quad M_{22} = \gamma \sqrt{P} \sqrt{S + 2\sqrt{P}}.$$

It is clear from (2.18), (2.19) and (2.13) that \mathbf{M} is Hermitian.

REMARK 1. It is clear from (2.13), (2.18) and (2.19) that M_{ik} are real and:

$$(2.20) \quad M_{11} > 0, \quad M_{22} > 0.$$

3. Equations for the H/V ratio

• Secular equation:

Consider a Rayleigh wave propagating in an incompressible deformed isotropic elastic half-space $x_2 \geq 0$, as described in Subsection 2.2, with the velocity $c(> 0)$, the wave number $k(> 0)$, in the x_1 -direction and decaying in the x_2 -direction. Suppose the half-space is free of traction, then the secular equation of the wave is [6, 11]:

$$(3.1) \quad \gamma(\alpha - X) + (2\beta + 2\gamma_* - X) \sqrt{\gamma(\alpha - X)} - \gamma_*^2 = 0.$$

In terms of the dimensionless parameters

$$(3.2) \quad \delta_1 = \gamma/\alpha (> 0), \quad \delta_2 = \beta/\alpha, \quad \delta_3 = \gamma_*/\alpha$$

the secular equation (3.1) becomes

$$(3.3) \quad \delta_1(1-x) + \sqrt{\delta_1}(2\delta_2 + 2\delta_3 - x)\sqrt{1-x} - \delta_3^2 = 0, \quad 0 < x < 1,$$

where $x = c^2/c_2^2$, $c_2 = \sqrt{\alpha/\rho}$, $0 < x < 1$ by (2.12).

• *Relation between the H/V ratio and the Rayleigh wave velocity:*

Let \mathbf{M} is the surface impedance matrix of the Rayleigh wave. Then it is given by (2.18), (2.19). Since the half-space is free of traction, i.e $\mathbf{\Sigma}(0) = \mathbf{0}$, it follows from (2.2): $\mathbf{M}\mathbf{U}(0) = \mathbf{0}$. From this equation and (2.18) we have:

$$(3.4) \quad \begin{cases} M_{11} \frac{U_1(0)}{U_2(0)} + iM_{12} = 0, \\ -iM_{12} \frac{U_1(0)}{U_2(0)} + M_{22} = 0. \end{cases}$$

From (3.4) it follows:

$$(3.5) \quad \left[\frac{U_1(0)}{U_2(0)} \right]^2 = -\frac{M_{22}}{M_{11}}.$$

Since $-M_{22}/M_{11} < 0$ by Remark 1, Eq. (3.5) provides:

$$(3.6) \quad \frac{U_1(0)}{U_2(0)} = i\sqrt{\frac{M_{22}}{M_{11}}}.$$

By the definition of the H/V ratio $\kappa = |U_1(0)/U_2(0)|$, thus we have

$$(3.7) \quad \kappa = \sqrt{\frac{M_{22}}{M_{11}}} \Rightarrow \kappa^2 = \frac{M_{22}}{M_{11}}.$$

Introducing the expressions of M_{11} and M_{22} given by (2.19) into the second of (3.7) we arrive at

$$(3.8) \quad \kappa^2 = \frac{\sqrt{1-x}}{\sqrt{\delta_1}}.$$

This is the desired relation between the H/V ratio and the Rayleigh wave velocity.

REMARK 2. Since $0 < x < 1$, it follows from (3.8): $0 < \kappa^2 < 1$ if $\alpha < \gamma$; κ^2 may go to ∞ if α is much larger than γ .

• *Equations for the H/V ratio:*

Putting $w = \kappa^2$, from (3.8) we have

$$(3.9) \quad w = \frac{\sqrt{1-x}}{\sqrt{\delta_1}}, \quad 0 < x < 1.$$

Eliminating x from (3.3) and (3.9) yields a cubic equation for w (provided $\delta_3 \neq 0$)

$$(3.10) \quad f_1(w) := w^3 + w^2 + a_1 w + a_0 = 0, \quad w \in (0, \delta_1^{-1/2}),$$

where

$$(3.11) \quad a_0 = -\frac{\delta_3^2}{\delta_1^2}, \quad a_1 = (2\delta_2 + 2\delta_3 - 1)/\delta_1.$$

Equation (3.10) is the equation determining the H/V ratio. *It is interesting that Eq. (3.10) for κ^2 is just Eq. (5.26) in [6] for η .*

If $\delta_3 = 0 \Rightarrow a_0 = 0$, then Eq. (3.10) is equivalent to a quadratic equation, namely

$$(3.12) \quad f_2(w) := w^2 + w + a_1 = 0, \quad w \in (0, \delta_1^{-1/2}).$$

By w_r we denote a root of Eq. (3.10) or (3.12) that belongs to the interval $(0, \delta_1^{-1/2})$.

• *Existence of solution of the H/V ratio equations:*

Since the mapping (3.9) is a 1-1 mapping that maps $x \in (0, 1)$ to $w \in (0, \delta_1^{-1/2})$, it follows that Eq. (3.10) has a unique solution (w_r) if and only if Eq. (3.3) has a unique solution ($\in (0, 1)$). From this fact and Propositions 1 and 2 in [11] (see also (6.9) and (6.17) in [6]) we have

PROPOSITION 1. *Suppose $\delta_3 \neq 0$, then Eq. (3.10) has a unique root w_r if and only if*

$$(3.13) \quad \sqrt{\delta_1} + 2\delta_2 + 2\delta_3 - \frac{\delta_3^2}{\sqrt{\delta_1}} > 0.$$

PROPOSITION 2. *Let $\delta_3 = 0$, then Eq. (3.12) has a unique root w_r if and only if*

$$(3.14) \quad -\sqrt{\delta_1} < 2\delta_2 < 1.$$

PROPOSITION 3. *Suppose $\delta_3 \neq 0$ and (3.13) holds. If Eq. (3.10) has two or three distinct real roots, then w_r is the largest root.*

Proof. Suppose $\delta_3 \neq 0$ and (3.13) holds and Eq. (3.10) has two or three distinct real roots, then: $\Delta' > 0$, where $\Delta' = 1 - 3a_1$ is the discriminant of

the equation $f_1'(w) = 3w^2 + 2w + a_1 = 0 \Rightarrow$ the equation $f_1'(w) = 0$ has two distinct roots, denoted by w_{max} and w_{min} so that either $w_{max} < w_{min} \leq 0$ or $w_{max} < 0 < w_{min}$ due to $w_{max} + w_{min} = -2/3 < 0$. If $w_{max} < w_{min} \leq 0$: because $f_1(0) < 0$, $f_1(+\infty) = +\infty$ and $f_1(w)$ is strictly monotonically increasing in $(0, +\infty)$, the equation $f_1(w) = 0$ has therefore a unique root in $(0, +\infty)$ and this root is w_r . If $w_{max} < 0 < w_{min}$: since $f_1(w)$ is strictly monotonically decreasing in (w_{max}, w_{min}) and $f_1(0) < 0$ it follows: $f_1(w_{min}) < 0$. As $f_1(w)$ is strictly monotonically increasing in $(w_{min}, +\infty)$, $f_1(w_{min}) < 0$ and $f_1(+\infty) = +\infty$, equation $f_1(w) = 0$ has therefore a unique root in $(w_{min}, +\infty)$ so in $(0, +\infty)$ and it is w_r . That means w_r is the largest real root.

4. Formulas for the H/V ratio for a general strain-energy function

THEOREM 1. *If there exists a Rayleigh wave propagating along the x_1 -direction, and attenuating in the x_2 -direction, in an incompressible isotropic elastic half-space subject to a homogeneous initial deformation (Eq. (2.3)), then it is unique, and its squared H/V ratio κ^2 is determined as follows:*

i) If $\delta_3 \neq 0$:

$$(4.1) \quad \kappa^2 = -\frac{1}{3} + \sqrt[3]{R + \sqrt{D}} + \frac{(1 - 3a_1)}{9\sqrt[3]{R + \sqrt{D}}},$$

where each radical is understood as the complex root taking its principal value, R and D are given by:

$$(4.2) \quad R = (9a_1 - 27a_0 - 2)/54, \quad D = (4a_0 - a_1^2 - 18a_0a_1 + 27a_0^2 + 4a_1^3)/108.$$

a_0 and a_1 are determined by (3.11).

ii) If $\delta_3 = 0$:

$$(4.3) \quad \kappa^2 = \frac{\sqrt{\delta_1 - 8\delta_2 + 4} - \sqrt{\delta_1}}{2\sqrt{\delta_1}}.$$

Proof. The uniqueness of Rayleigh waves follows immediately from Propositions 1 and 2. Now we present the derivation of the formulas (4.1) and (4.3).

(i) Suppose $\delta_3 \neq 0$ and (3.13) holds. Then, a unique Rayleigh wave can propagate in the half-space, according to Proposition 1, and its H/V ratio is determined by Eq. (3.10). Let $z = w + 1/3$, then in terms of z Eq. (3.10) has the form:

$$(4.4) \quad z^3 - 3q^2z + r = 0,$$

where

$$(4.5) \quad r = -2R, \quad q^2 = \frac{(a_2^2 - 3a_1)}{9}.$$

According to the theory of cubic equations, three roots z_k ($k = 1, 2, 3$) of Eq. (4.4) are calculated by [21]:

$$(4.6) \quad \begin{aligned} z_1 &= S + T, & z_2 &= -\frac{1}{2}(S + T) + \frac{i\sqrt{3}}{2}(S - T), \\ z_3 &= -\frac{1}{2}(S + T) - \frac{i\sqrt{3}}{2}(S - T), \end{aligned}$$

where

$$(4.7) \quad S = \sqrt[3]{R + \sqrt{D}}, \quad T = \sqrt[3]{R - \sqrt{D}}, \quad D = R^2 + Q^3, \quad Q = -q^2.$$

In relation to the formulas (4.7) we emphasize two points:

- The cube root of a negative real number is taken as the real negative root.
- If, in the expression S , $R + \sqrt{D}$ is complex, the phase angle in T is taken as the negative of the phase angle in S so that $T = S^*$ where S^* is the complex conjugate of S .

REMARK 3. If $D > 0$, then Eq. (4.4) has one real root and two complex conjugate roots.

If $D = 0$, this equation has three real roots, at least two of which are equal.

If $D < 0$, it has three real distinct roots.

Let $z_r = 1/3 + w_r$, then z_r is a real root of Eq. (4.4) and if Eq. (4.4) has two or three real roots, z_r is the largest real root, according to Proposition 3. We prove that z_r is given by

$$(4.8) \quad z_r = \sqrt[3]{R + \sqrt{D}} + \frac{q^2}{\sqrt[3]{R + \sqrt{D}}},$$

where each radical is understood as a complex root taking its principal value, R and D are calculated by (4.2), q^2 is given by (4.5)₂. Formula (4.1) is obtained immediately from (4.8) and the relation $w_r = -1/3 + z_r$. We consider the distinct cases dependent on the values of D for proving (4.8).

- For the values of $D > 0$, according to Remark 3, Eq. (4.4) has a unique real root, so it is z_r , given by (4.6)₁:

$$(4.9) \quad z_r = \sqrt[3]{R + \sqrt{D}} + \sqrt[3]{R - \sqrt{D}},$$

in which the radicals are understood as real ones. To prove (4.8) we have to show that the right side of (4.9) in which the radicals is understood as real ones coincides with the right side of (4.8) where each radical being understood as a complex root taking its principal value. Since

$$(4.10) \quad \sqrt[3]{R - \sqrt{D}} = \frac{q^2}{\sqrt[3]{R + \sqrt{D}}}$$

it is sufficient to prove that $R + \sqrt{D} > 0$. Note that, since Eq. (4.4) has a unique real root, so does Eq. (3.10). To prove $R + \sqrt{D} > 0$ we examine the distinct cases dependent on the values of Δ' , the discriminant of the equation $f_1'(w) = 0$.

If $\Delta' \leq 0$, then $f_1(w)$ is strictly monotonically increasing in $(-\infty, +\infty)$. By w_N we denote the abscissa of the point of inflexion N of the cubic curve $y = f_1(w)$, then $w_N = -2/3 < 0$. This and the fact $f_1(0) = a_0 < 0$ and the strictly increasing monotonousness of $f_1(w)$ lead to $f_1(w_N) < 0$. Since $r = f_1(w_N)$ it follows $r < 0$, or equivalently $R > 0$. This leads to $R + \sqrt{D} > 0$.

If $\Delta' > 0 \Rightarrow f_1'(w) = 0$ has two distinct roots w_{max} and w_{min} and either $w_{max} < w_{min} \leq 0$ or $w_{max} < 0 < w_{min}$ (see Proposition 3). In both two cases we always have: $f_1(w_{min}) < 0$. As Eq. (3.10) has a unique real root as addressed above it follows: $f_1(w_{max})f_1(w_{min}) > 0$, consequently, $f_1(w_{max}) < 0$. This and $f_1(w_{min}) < 0$ provide $r = f_1(w_N) < 0 \Rightarrow R = -r/2 > 0$, therefore we have $R + \sqrt{D} > 0$.

- For $D = 0$, analogously as above, one can see that $r < 0$, consequently $R > 0$. When $D = 0$ we have $R^2 = -Q^3 = q^6$ ($q > 0$) $\Rightarrow R = q^3 \Rightarrow r = -2R = -2q^3$, so Eq. (4.4) becomes $z^3 - 3q^2z - 2q^3 = 0$ whose roots are: $z_1 = 2q$, $z_2 = -q$ (double root). This says $z_r = 2q$, since it is the largest root. With the help of $q > 0$ and $D = 0$ it is readily seen that z_r calculated by (4.8) is $2q$.

- For the values of $D < 0$, according to Remark 3, Eq. (4.4) has three distinct real roots and z_r is the largest one. Using the arguments presented in [22], p. 255, it is not difficult to verify that in this case the largest real root of Eq. (4.4) is:

$$(4.11) \quad z_r = \sqrt[3]{R + \sqrt{D}} + \sqrt[3]{R - \sqrt{D}}$$

in which each radical is understood as a complex root taking its principal value. By 3θ we denote the phase angle of $R + i\sqrt{-D}$. Then, it is not difficult to prove that:

$$(4.12) \quad \sqrt[3]{R + \sqrt{D}} = qe^{i\theta}, \quad \sqrt[3]{R - \sqrt{D}} = qe^{-i\theta}$$

where radicals are understood as complex roots taking their principal value. From (4.12) we have immediately (4.10) and then (4.8) by taking into account (4.11).

(ii) Let $\delta_3 = 0$. According to Proposition 2, a unique Rayleigh wave can propagate in the half-space if and only if (3.14) holds and the H/V ratio is computed by Eq. (3.12). One can see that when (3.14) is valid, the quadratic equation (3.12): $f_2(w) = 0$ has two distinct real roots w_1 and w_2 so that: $w_1 < 0 < w_2$ and $w_2 = w_r$. It is readily to verify that w_2 , so w_r , is calculated by the formula (4.3).

The proof of Theorem 1 is completed.

5. Formulas of the H/V for specific strain-energy functions

5.1. The neo-Hookean material

For this material the strain-energy function is of the form [6]:

$$(5.1) \quad W(\lambda_1, \lambda_2) = \frac{1}{2}\mu \left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3 \right),$$

where μ is shear modulus. Consider the plain strain with $\lambda_3 = 1$. From (2.8), (2.9) and (5.1) we have:

$$(5.2) \quad \alpha = \mu\lambda^2, \quad \gamma = \frac{\mu}{\lambda^2}, \quad \beta = \frac{\mu}{2}(\lambda^2 + \frac{1}{\lambda^2}), \quad \gamma_* = \frac{\mu}{\lambda^2} - \sigma_2,$$

thus, according to (3.2) and (5.2)

$$(5.3) \quad \delta_1 = \frac{1}{\lambda^4}, \quad \delta_2 = \frac{1}{2} \left(1 + \frac{1}{\lambda^4} \right), \quad \delta_3 = \frac{1 - \lambda^2 \bar{\sigma}_2}{\lambda^4},$$

where $\lambda := \lambda_1$, $\bar{\sigma}_2 = \sigma_2/\mu$. Note that $\lambda_2 = \lambda^{-1}$ due to the incompressibility condition $\lambda_1 \lambda_2 \lambda_3 = 1$ and $\lambda_3 = 1$. From (3.11) and (5.3), the coefficients a_0 and a_1 of the H/V ratio equation (3.10) are

$$(5.4) \quad a_1 = 3 - 2\lambda^2 \bar{\sigma}_2, \quad a_0 = -(1 - \lambda^2 \bar{\sigma}_2)^2.$$

According to Proposition 1 and Theorem 1(i), if $\delta_3 \neq 0$, i. e. $(1 - \lambda^2 \bar{\sigma}_2) \neq 0$, then a Rayleigh wave is possible if and only if (coming from (3.13) and (5.3)):

$$(5.5) \quad (1 + \lambda - \lambda^2 + \lambda^3 - \lambda^2 \bar{\sigma}_2)(-1 + \lambda + \lambda^2 + \lambda^3 + \lambda^2 \bar{\sigma}_2) > 0$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by (5.4)₁, R and D are calculated by:

$$(5.6) \quad R = \frac{26}{27} + \frac{\lambda^2 \bar{\sigma}_2 (-8 + 3\lambda^2 \bar{\sigma}_2)}{6}, \quad D = \frac{(-2 + \lambda^2 \bar{\sigma}_2)^2 (44 - 68\lambda^2 \bar{\sigma}_2 + 27\lambda^4 \bar{\sigma}_2^2)}{108}.$$

From (5.3) it is readily seen that $2\delta_2 > 1$, the condition (3.14) is therefore not satisfied. According to Proposition 2, a Rayleigh wave is impossible for the case $\delta_3 = 0$, i.e.,

$$(5.7) \quad 1 - \lambda^2 \bar{\sigma}_2 = 0.$$

Thus, the H/V ratio is not defined at points in the space of λ and $\bar{\sigma}_2$ satisfying (5.7). When $\bar{\sigma}_2 = 0$, from (5.4) and (5.6) it follows:

$$(5.8) \quad a_1 = 3, \quad R = \frac{26}{27}, \quad D = \frac{44}{27}$$

and using (4.1) and (5.8) gives $w_r = 0.2956$. The condition (5.5) for this case is $\lambda > 0.5437$.

Figure 1 shows some contour lines of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ (shaded) given by (5.5). The thick continuous curve is the set of points, defined by Eq. (5.7), at which the Rayleigh wave does not exist and hence the corresponding H/V ratio is not defined. The H/V ratio tends to zero when $(\lambda, \bar{\sigma}_2)$ approaching this curve. If λ is fixed and $\bar{\sigma}_2$ varies in its possible range, the squared H/V ratio approaches the supremum value $\delta_1^{-1/2} = \lambda^2$ at the two end points.

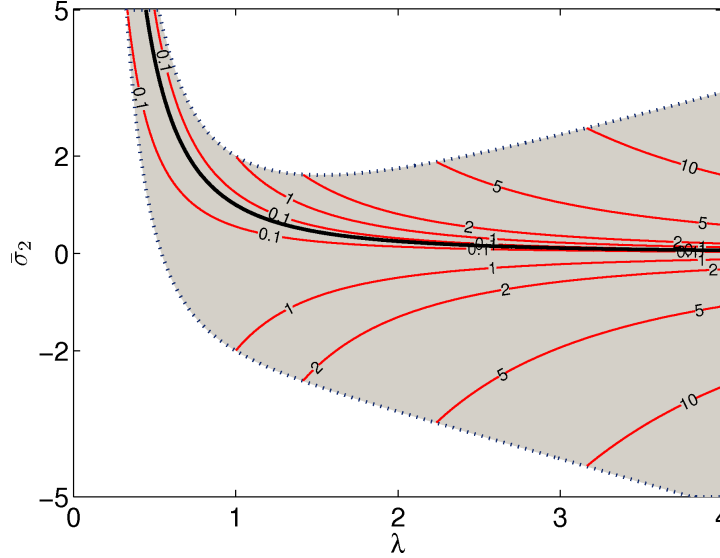


FIG. 1. Some contours of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ (shaded) for the neo-Hookean material.

Figure 2 shows the dependence of the H/V ratio on λ and $\bar{\sigma}_2$ computed by the exact formula (4.1) (along with (5.4), (5.6)). The up figure shows the dependence of the H/V ratio on λ with two fixed values of $\bar{\sigma}_2$ being 0 and 1. Each curve starts from a so-called cut-off value of λ computed by (5.5). When $\bar{\sigma}_2 = 0$, the H/V ratio is independent of λ and equals 0.2956, as mentioned above. The cut-off value of λ in this case is 0.5437. For $\bar{\sigma}_2 = 1$ the cut-off value of λ is 0.4656 and the squared H/V ratio starts from 0.2168 which is the square of 0.4656. For the case $\bar{\sigma}_2 = 1$ the exact curve first decreases and approaches zero when λ tends to $\lambda_0 = 1$ (the root of Eq. (5.7) with $\bar{\sigma}_2 = 1$) and then increases with λ .

The down figure shows the dependence of the H/V ratio on $\bar{\sigma}_2$ with two fixed values of λ : $\lambda = 1$ and $\lambda = 2$. The range of $\bar{\sigma}_2$ for the case $\lambda = 1$ is

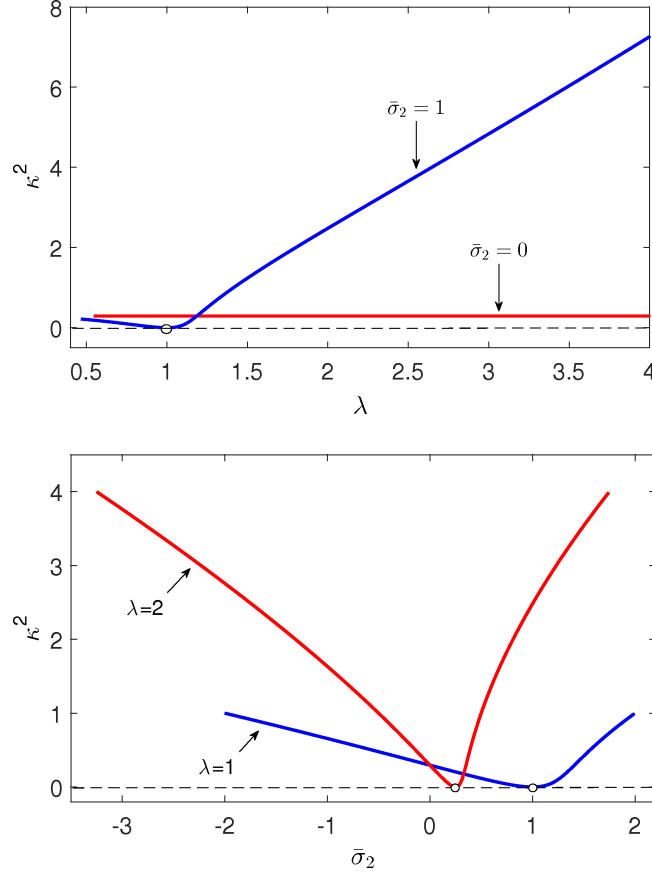


FIG. 2. The H/V ratio curves computed by exact formula as a function of λ (the up figure) and $\bar{\sigma}_2$ (the down figure) with $\lambda_3 = 1$ for the neo-Hookean material.

from -2 to 2 and it is from -3.25 to 1.75 for $\lambda = 2$. The H/V ratio curves take a "V" shape with two supremum points $\delta_1^{-1/2} = \lambda^2$ at the two end points, which are 1 and 4 for $\lambda = 1$ and 2 , respectively. They have an infimum point: at $\bar{\sigma}_2 = 1$ for $\lambda = 1$ and at $\bar{\sigma}_2 = 0.25$ for $\lambda = 2$, following (5.7). The H/V ratio tends to zero when $\bar{\sigma}_2$ approaching 1 (for the case $\lambda = 1$) and 0.25 (for the case $\lambda = 2$).

5.2. The Varga material

For the Varga material, the strain-energy function takes the form [6]:

$$(5.9) \quad W(\lambda_1, \lambda_2) = 2\mu(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1\lambda_2} - 3).$$

Consider $\lambda_3 = 1$ and denote $\lambda_1 = \lambda$, from (2.8), (2.9) and (5.9) we have

$$(5.10) \quad \begin{aligned} \alpha &= 2\mu \frac{\lambda^3}{1 + \lambda^2}, & \gamma &= 2\mu \frac{1}{\lambda(1 + \lambda^2)}, \\ \beta &= 2\mu \frac{\lambda}{1 + \lambda^2}, & \gamma_* &= 2\mu \frac{1}{\lambda(1 + \lambda^2)} - \sigma_2. \end{aligned}$$

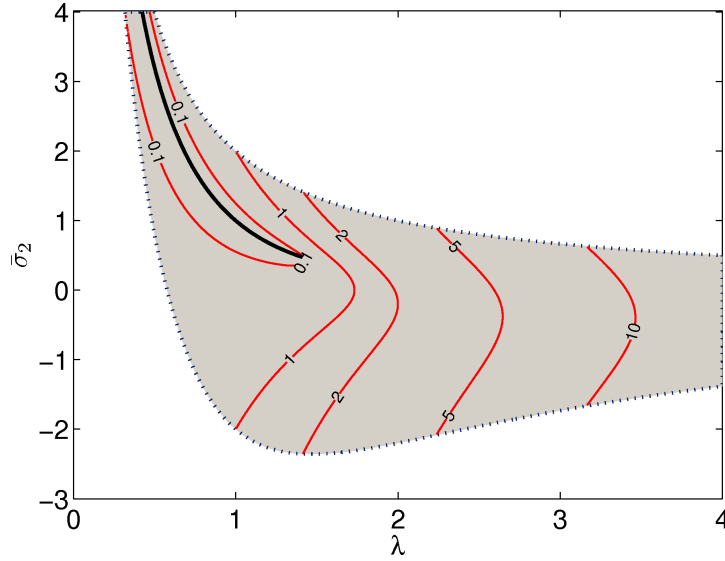


FIG. 3. Some contours of the H/V-ratio in the possible space of λ and $\bar{\sigma}_2$ (shaded) for the Varga material with $\lambda_3 = 1$.

From (3.2) and (5.10) it follows

$$(5.11) \quad \delta_1 = \frac{1}{\lambda^4}, \quad \delta_2 = \frac{1}{\lambda^2}, \quad \delta_3 = \frac{2 - \lambda(1 + \lambda^2)\bar{\sigma}_2}{2\lambda^4}.$$

From (3.11) and (5.11), the coefficients a_0 and a_1 of the H/V ratio equation (3.10) are

$$(5.12) \quad a_1 = -(\lambda^4 + \lambda^3\bar{\sigma}_2 - 2\lambda^2 + \lambda\bar{\sigma}_2 - 2), \quad a_0 = -\left[\frac{\lambda(1 + \lambda^2)\bar{\sigma}_2 - 2}{2}\right]^2.$$

According to Proposition 1 and Theorem 1(i), if $\delta_3 \neq 0$, i. e. $[2 - \lambda(1 + \lambda^2)\bar{\sigma}_2] \neq 0$, then a Rayleigh wave exists if and only if (following from (3.13) and (5.10)):

$$(5.13) \quad (2 - \lambda\bar{\sigma}_2)(\lambda^3\bar{\sigma}_2 + 6\lambda^2 + \lambda\bar{\sigma}_2 - 2) > 0$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by (5.12)₁, R and D are calculated by (4.2) and (5.12). If $\delta_3 = 0$, i.e.,

$$(5.14) \quad \lambda(1 + \lambda^2)\bar{\sigma}_2 - 2 = 0,$$

then a Rayleigh wave can propagate in the half-space if and only if (originating from (3.14) and (5.10))

$$(5.15) \quad \lambda > \sqrt{2}$$

and from (4.3) the H/V ratio of the Rayleigh wave is given by

$$(5.16) \quad w_r := \kappa^2 = -\frac{1}{2} + \frac{1}{2}\sqrt{4\lambda^4 - 8\lambda^2 + 1}, \quad \lambda > \sqrt{2}.$$

When $\bar{\sigma}_2 = 0$, from (5.11) and (5.12) we have

$$(5.17) \quad \delta_3 = \frac{1}{\lambda^4} > 0, \quad a_0 = -1, \quad a_1 = 2 + 2\lambda^2 - \lambda^4,$$

therefore by Proposition 1 and Theorem 1(i), a Rayleigh wave is possible if and only if $\lambda > 1/\sqrt{3}$ (following from (5.13) with $\bar{\sigma}_2 = 0$) and the H/V ratio is given by (4.1) in which a_1 is given by (5.17)₃ and

$$(5.18) \quad \begin{aligned} R &= \frac{43}{54} + \frac{1}{3}\lambda^2 - \frac{1}{6}\lambda^4, \\ D &= -(\lambda^2 - 3)(\lambda^2 + 1)(4\lambda^8 - 16\lambda^6 + 5\lambda^4 + 22\lambda^2 + 29)/108. \end{aligned}$$

Unlike the neo-Hookean material, the H/V ratio depends on λ for the case $\bar{\sigma}_2 = 0$.

Figure 3 shows some contour lines of the squared H/V ratio in the possible domain of λ and $\bar{\sigma}_2$ defined by the condition given in (5.13). When $(\lambda, \bar{\sigma}_2)$ approaching the boundary of this domain, the squared H/V ratio tends to λ^2 . The thick continuous curve is expressed by Eq. (5.14) with $0 < \lambda \leq \sqrt{2}$. The H/V ratio goes to zero when $(\lambda, \bar{\sigma}_2)$ approaching this curve.

The up (down) figure in Fig. 4 shows the dependence of the H/V ratio on λ ($\bar{\sigma}_2$) with two fixed values of $\bar{\sigma}_2 = 0; 1$ (of $\lambda = 1; 2$) that is calculated by the exact formula (4.1) (along with (4.2)). In the up Fig. 4, for $\bar{\sigma}_2 = 0$, the cut-off value of λ is 0.5773 determined by (5.13), for the case $\bar{\sigma}_2 = 1$, the range of λ is (0.4836, 2). In the down Fig. 4, the range of $\bar{\sigma}_2$ is $(-2, 2)$ for $\lambda = 1$ and $(-11/5, 1)$ for $\lambda = 2$ and at the end points, the squared H/V ratio approaches λ^2 . On both figures, there is a point (marked by circles) with $\bar{\sigma}_2 = 1$ and $\lambda = 1$ at which the Rayleigh waves do not exist. This point belongs to the thick curve shown in Fig. 3.

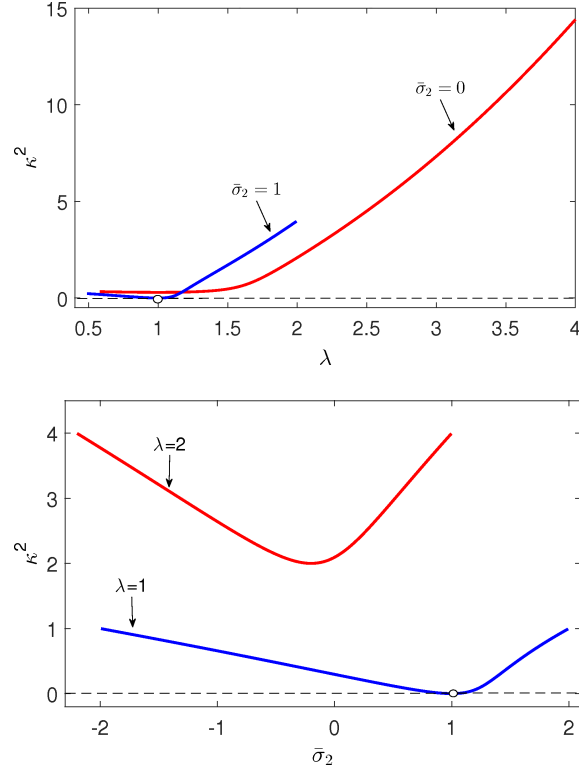


FIG. 4. The squared H/V ratio computed by exact formula as a function of λ (the up figure) and $\bar{\sigma}_2$ (the down figure) with $\lambda_3 = 1$ for incompressible Varga material.

5.3. The $m = 1/2$ material

For the $m = 1/2$ material, the strain-energy function is of the form [6]

$$(5.19) \quad W(\lambda_1, \lambda_2) = 8\mu \left(\lambda_1^{1/2} + \lambda_2^{1/2} + \frac{1}{\lambda_1^{1/2} \lambda_2^{1/2}} - 3 \right).$$

Consider $\lambda_3 = 1$ and denote $\lambda_1 = \lambda$, from (2.8), (2.9) and (5.19) we have:

$$(5.20) \quad \begin{aligned} \alpha &= \frac{4\mu\lambda^4}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, & \gamma &= \frac{4\mu}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, \\ \beta &= \frac{\mu(-\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda - 1)}{\sqrt{\lambda}(\lambda+1)(\lambda^2+1)}, & \gamma_* &= \gamma - \sigma_2. \end{aligned}$$

From (3.2) and (5.20) it follows:

$$(5.21) \quad \begin{aligned} \delta_1 &= \frac{1}{\lambda^4}, & \delta_2 &= -\frac{1}{4} + \frac{1}{2\lambda} + \frac{1}{2\lambda^2} + \frac{1}{2\lambda^3} - \frac{1}{4\lambda^4}, \\ \delta_3 &= \frac{4 - \sqrt{\lambda}(\lambda + 1)(\lambda^2 + 1)\bar{\sigma}_2}{4\lambda^4}. \end{aligned}$$

From (3.11) and (5.21), the coefficients of the H/V ratio equation (3.10) are

$$(5.22) \quad \begin{aligned} a_1 &= (1 + \lambda)(1 + \lambda^2)(1 - \sqrt{\lambda}\bar{\sigma}_2/2) + \frac{1 - 3\lambda^4}{2}, \\ a_0 &= -\left(\frac{\sqrt{\lambda}(1 + \lambda)(1 + \lambda^2)\bar{\sigma}_2}{4} - 1\right)^2. \end{aligned}$$

According to Proposition 1 and Theorem 1(i), if $\delta_3 \neq 0$, i. e. $[4 - \sqrt{\lambda}(\lambda + 1)(\lambda^2 + 1)\bar{\sigma}_2] \neq 0$, then a Rayleigh wave exists if and only if (following from (3.13) and (5.21))

$$(5.23) \quad 8(\lambda^2 - 4\lambda + 2) + 8\sqrt{\lambda}(\lambda - 1)\bar{\sigma}_2 + \lambda(\lambda^2 + 1)\bar{\sigma}_2^2 < 0,$$

and the H/V ratio of the Rayleigh wave is calculated by the formula (4.1) in which a_1 is given by (5.22)₁, R and D are calculated by (4.2) and (5.22).

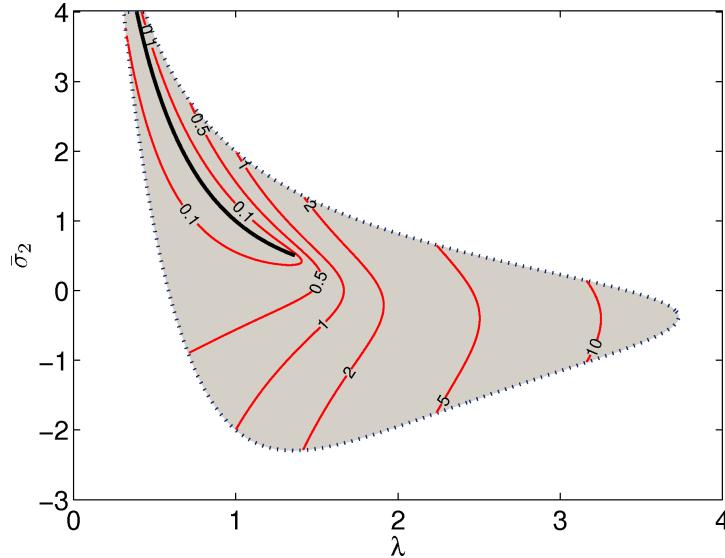


FIG. 5. Some contours of different values of squared H/V-ratio in space of λ and $\bar{\sigma}_2$ for incompressible $m = 1/2$ material with $\lambda_3 = 1$.

In case $\bar{\sigma}_2 = 0$, the equation of the H/V ratio becomes

$$(5.24) \quad w^3 + w^2 + \left(\frac{3}{2} + \lambda + \lambda^2 + \lambda^3 - \frac{3}{2}\lambda^4 \right) w - 1 = 0$$

and its solution depends on the principal stresses λ , unlike the case of Neo-Hookean's material.

Figure 5 shows some contour lines of squared H/V ratio in the domain of λ and $\bar{\sigma}_2$ in which the Rayleigh surface waves exist. Unlike the Neo-Hookean and Varga materials, this domain is bounded in λ . The picture of contour lines in this material is similar to that of Varga material. The thick continuous curve shows the set of points at which H/V ratio is not defined. In this case, $\lambda < 1.3756$, and H/V ratio approaches to zero around this curve.

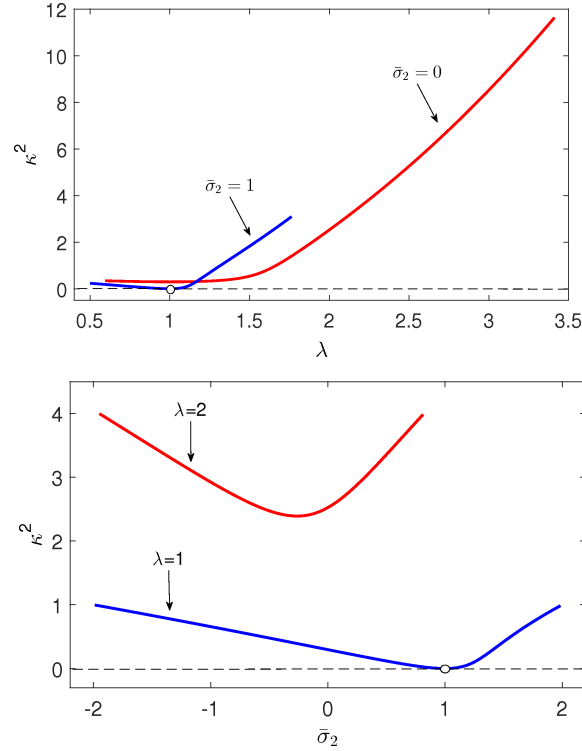


FIG. 6. The squared H/V ratio computed by exact formula as a function of λ (the up figure) and $\bar{\sigma}_2$ (the down figure) with $\lambda_3 = 1$ for incompressible $m = 1/2$ material.

Figure 6 shows the dependence of the squared H/V ratio on λ (and $\bar{\sigma}_2$) using (4.1) and (5.6). In the up figure, for $\bar{\sigma}_2 = 0$, Rayleigh waves exist in $2 - \sqrt{2} < \lambda < 2 + \sqrt{2}$. For $\bar{\sigma}_2 = 1$, λ varies from 0.4896 to 1.7734. The range of $\bar{\sigma}_2$ on the down figure is $(-2, 2)$ and $(\frac{2}{5}(-\sqrt{2} - 2\sqrt{3}), \frac{2}{5}(-\sqrt{2} + 2\sqrt{3}))$ for λ equals 1 and 2, respectively.

6. Conclusions

In this paper, the exact H/V ratio formulas have been derived by solving analytically the H/V ratio equations. These formulas are valid for a general strain-energy function. Several particular strain-energy functions are employed to specify these formulas. Some numerical examples are carried out to examine the dependence of the H/V ratio on the pre-stress. Since the H/V ratio is a convenient tool for nondestructively evaluating pre-stresses of structures before and during loading, the obtained formulas will be significant in practical applications.

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