

Screw dislocations in piezoelectric laminates with four or more phases

X. WANG^{1,*}, L. CHEN¹, P. SCHIAVONE^{2,*}

¹*School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai 200237, China,
e-mail: xuwang@ecust.edu.cn

²*Department of Mechanical Engineering, University of Alberta, 10-203 Donadeo Innovation Centre for Engineering, Edmonton, Alberta Canada T6G 1H9,
e-mail: p.schiavone@ualberta.ca

WE PRESENT AN ANALYTICAL SOLUTION TO THE PROBLEM of a screw dislocation in a four-phase piezoelectric laminate composed of two piezoelectric layers of equal thickness sandwiched between two semi-infinite piezoelectric media. A new version of the complex variable formulation is proposed such that the 2×2 real symmetric matrix appearing in the formulation becomes dimensionless. Using analytic continuation, the original boundary value problem is reduced to the identification of a single 2D analytic vector function which is completely determined following rigorous solution of the resulting linear recurrence relations in matrix form. An explicit expression for the image force acting on the piezoelectric screw dislocation is obtained once the single 2×2 real matrix function is identified. We also discuss the solution for a screw dislocation in an N -phase piezoelectric laminate composed of $N - 2$ piezoelectric layers of equal thickness sandwiched between two semi-infinite piezoelectric media.

Key words: four-phase piezoelectric laminate, screw dislocation, analytical solution, analytic continuation, recurrence relations of matrix form.

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1. Introduction

STUDIES OF DISLOCATIONS IN MULTILAYERED MEDIA are extremely challenging mainly because of the presence of multiple interfaces and surfaces (see, for example, [1–9]). Even more difficulties are encountered if this type of interaction problem is discussed in the context of piezoelectric laminated materials exhibiting the well-known electromechanical coupling phenomenon. This may explain why theoretical investigations into the influence of dislocations in multilayered piezoelectric laminates are few and far between. Most recently, however, using the technique of analytic continuation, WANG and SCHIAVONE [10] derived analytical solutions for a screw dislocation or a two-dimensional Eshelby

inclusion of arbitrary shape in a three-phase piezoelectric laminate composed of two semi-infinite piezoelectric media bonded together via an intermediate piezoelectric layer of finite thickness. Given the importance of layered models in the design and optimization of piezoelectric devices (Layered geometry enjoys high sensitivity, excellent bandwidth and enhanced reception characteristics: see for example, GUO *et al.* [11] and the references therein. Related studies on multilayered piezoelectric laminates (which are used widely in smart structure design) with the objectives of active shape control, structural vibration and noise control can also be found in [12–14]), it is of great interest to extend this most recent study to piezoelectric laminates containing four or more phases. We can see from the present investigation that this extension is highly non-trivial and quite challenging.

In this paper, we consider the interaction problem associated with a screw dislocation in a four-phase piezoelectric laminate composed of two piezoelectric layers of equal thickness encased in two semi-infinite piezoelectric media. The screw dislocation, which suffers discontinuities in anti-plane displacement and in electric potential across the slip plane and which is also subjected to a line force and a line charge at its core, is located in the upper semi-infinite piezoelectric medium. A full-field analytical solution is derived using: (i) a new version of the complex variable formulation involving a 2×2 dimensionless real symmetric matrix; (ii) analytic continuation [15, 16]; (iii) a rigorous solution of the resulting linear recurrence relations in matrix form. A concise and elegant expression for the image force acting on the screw dislocation containing a single 2×2 real matrix function is obtained using the extended version of the Peach–Koehler formula [17, 18]. It is shown via a specific example that the proposed theory can be implemented quite expediently. Finally, we also obtain the solution structure for a screw dislocation in an N -phase piezoelectric laminate composed of $N - 2$ piezoelectric layers of equal thickness encased in two semi-infinite piezoelectric media. Here N is an integer equal to or greater than 5. We also obtain the image force acting on the screw dislocation which is found to be completely controlled by a single 2×2 real matrix function. Specific results are presented for each of the two cases of $N = 5, 6$ to demonstrate the derived general solution.

2. New version of the complex variable formulation

For the anti-plane shear deformations of a hexagonal piezoelectric material exhibiting 6 mm symmetry with its poling direction along the x_3 -axis, we propose the following new version of the complex variable formulation:

$$(2.1) \quad \left[\frac{u_3}{\phi \sqrt{\epsilon_0/C_0}} \right] = \text{Im}\{\mathbf{f}(z)\},$$

$$(2.2) \quad \begin{bmatrix} 2\varepsilon_{32} + 2i\varepsilon_{31} \\ -(E_2 + iE_1)\sqrt{\epsilon_0/C_0} \end{bmatrix} = \mathbf{f}'(z), \quad \begin{bmatrix} (\sigma_{32} + i\sigma_{31})/C_0 \\ (D_2 + iD_1)/\sqrt{C_0\epsilon_0} \end{bmatrix} = \mathbf{C}\mathbf{f}'(z),$$

where u_3 and ϕ are the anti-plane displacement and electric potential, respectively; σ_{31} and σ_{32} are the Cartesian anti-plane shear stresses; D_1 and D_2 are the electric displacements; E_1 and E_2 are the in-plane electric fields; ε_{31} and ε_{32} are strain components; C_0 is a reference stiffness and ϵ_0 is a reference dielectric constant; $\mathbf{f}(z)$ is a 2D analytic vector function of the complex variable $z = x_1 + ix_2$; the 2×2 dimensionless real symmetric matrix \mathbf{C} is defined by

$$(2.3) \quad \mathbf{C} = \mathbf{C}^T = \begin{bmatrix} \frac{C_{44}}{C_0} & \frac{e_{15}}{\sqrt{C_0\epsilon_0}} \\ \frac{e_{15}}{\sqrt{C_0\epsilon_0}} & -\frac{\epsilon_{11}}{\epsilon_0} \end{bmatrix}.$$

where C_{44} , e_{15} and ϵ_{11} are the elastic stiffness, the piezoelectric constant and the dielectric constant, respectively.

It is seen that the left-hand side of Eq. (2.1) has the dimension of length, whilst the left-hand sides of the two expressions in Eq. (2.2) are both dimensionless. In addition, C_0 and ϵ_0 are taken to be the same for different phases.

3. Analytical solution

Consider an entire z -plane composed of four piezoelectric phases:

$$S_1 : \text{Im}\{z\} \leq 0, \quad S_2 : 0 \leq \text{Im}\{z\} \leq h, \quad S_3 : h \leq \text{Im}\{z\} \leq 2h, \quad S_4 : \text{Im}\{z\} \geq 2h,$$

where $h (> 0)$ is the common thickness for each of the two intermediate layers. Each piezoelectric phase is hexagonal with its poling direction along the x_3 -axis. All of the four phases are perfectly bonded across the three planar interfaces. The piezoelectric composite is subjected to a screw dislocation at $z = id$, $d > 2h$ in S_4 . The screw dislocation suffers a jump b_3 in anti-plane displacement and a jump $\Delta\phi$ in electric potential across the slip plane. Meanwhile, it is subjected to a line force p and a line charge q at its core. Throughout what follows, the subscripts 1, 2, 3 and 4 are used to identify the respective quantities in S_1 , S_2 , S_3 and S_4 .

The boundary value problem for the four-phase piezoelectric laminate takes the form

$$(3.1) \quad \begin{aligned} \mathbf{f}_{j+1}(z) + \overline{\mathbf{f}_{j+1}(z)} &= \mathbf{\Gamma}_j \mathbf{f}_j(z) + \mathbf{\Gamma}_j \overline{\mathbf{f}_j(z)}, \\ \mathbf{f}_{j+1}(z) - \overline{\mathbf{f}_{j+1}(z)} &= \mathbf{f}_j(z) - \overline{\mathbf{f}_j(z)}, \quad \text{Im}\{z\} = h(j-1), \quad j = 1, 2, 3; \end{aligned}$$

$$(3.2) \quad \mathbf{f}_4(z) \cong \frac{1}{2\pi} (\hat{\mathbf{b}} - i\mathbf{C}_4^{-1}\hat{\mathbf{f}}) \ln(z - id) + O(1), \quad z \rightarrow id,$$

where

$$(3.3) \quad \mathbf{\Gamma}_j = \mathbf{C}_{j+1}^{-1} \mathbf{C}_j,$$

$$(3.4) \quad \hat{\mathbf{b}} = \left[b_3 \sqrt{\frac{\epsilon_0}{C_0}} \Delta\phi \right]^T, \quad \hat{\mathbf{f}} = \left[\frac{p}{C_0} - \frac{q}{\sqrt{C_0 \epsilon_0}} \right]^T.$$

It is seen from the above definition that both $\hat{\mathbf{b}}$ and $\hat{\mathbf{f}}$ have the dimension of length. It is assumed that the analytic vector function $\mathbf{f}_1(z)$ defined in the lower semi-infinite piezoelectric medium takes the following general form:

$$(3.5) \quad \mathbf{f}_1(z) = \mathbf{\Omega}(z), \quad \text{Im}\{z\} \leq 0,$$

where $\mathbf{\Omega}(z)$ is to be determined.

By enforcing the continuity conditions of displacement, electric potential, traction and normal electric displacement across the three perfect interfaces in Eq. (3.1), we arrive at the following general expressions for $\mathbf{f}_2(z)$, $\mathbf{f}_3(z)$ and $\mathbf{f}_4(z)$

$$(3.6) \quad \mathbf{f}_2(z) = \frac{1}{2}(\mathbf{\Gamma}_1 + \mathbf{I})\mathbf{\Omega}(z) + \frac{1}{2}(\mathbf{\Gamma}_1 - \mathbf{I})\bar{\mathbf{\Omega}}(z), \quad 0 \leq \text{Im}\{z\} \leq h;$$

$$(3.7) \quad \begin{aligned} \mathbf{f}_3(z) = & \frac{1}{4}(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I})\mathbf{\Omega}(z) + \frac{1}{4}(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})\mathbf{\Omega}(z - 2ih) \\ & + \frac{1}{4}(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})\bar{\mathbf{\Omega}}(z) \\ & + \frac{1}{4}(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I})\bar{\mathbf{\Omega}}(z - 2ih), \quad h \leq \text{Im}\{z\} \leq 2h; \end{aligned}$$

$$(3.8) \quad \begin{aligned} \mathbf{f}_4(z) = & \frac{1}{8}(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I})\mathbf{\Omega}(z) \\ & + \frac{1}{8}[(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ & + (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I})]\mathbf{\Omega}(z - 2ih) \\ & + \frac{1}{8}(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})\mathbf{\Omega}(z - 4ih) \\ & + \frac{1}{8}(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})\bar{\mathbf{\Omega}}(z) \\ & + \frac{1}{8}[(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ & + (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})]\bar{\mathbf{\Omega}}(z - 2ih) \\ & + \frac{1}{8}(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I})\bar{\mathbf{\Omega}}(z - 4ih), \quad \text{Im}\{z\} \geq 2h, \end{aligned}$$

where \mathbf{I} is a 2×2 identity matrix. The technique of analytic continuation [15, 16] has been applied in deriving Eqs. (3.6)–(3.8).

An examination of Eq. (3.8) suggests that $\mathbf{\Omega}(z)$ should take the following specified form

$$(3.9) \quad \mathbf{\Omega}(z) = \frac{4}{\pi} \sum_{j=1}^{+\infty} \mathbf{Q}_j (\hat{\mathbf{b}} - i\mathbf{C}_4^{-1}\hat{\mathbf{f}}) \ln[z - i(d + 2h(j-1))],$$

where \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ are 2×2 constant matrices to be determined.

Substituting the above expression into Eq. (3.8) and noting that $\mathbf{f}_4(z)$ exhibits the logarithmic singularity at $z = id$ in Eq. (3.2) and is regular at the points $z = i(d + 2jh)$, $j = 1, 2, \dots, \infty$, we obtain the following linear recurrence relations in matrix form:

$$(3.10) \quad \mathbf{A}_1 \mathbf{Q}_{3+m} + \mathbf{A}_2 \mathbf{Q}_{2+m} + \mathbf{A}_3 \mathbf{Q}_{1+m} = \mathbf{0}, \quad m = 0, 1, 2, \dots, \infty,$$

$$(3.11) \quad \begin{aligned} \mathbf{A}_1 \mathbf{Q}_2 + \mathbf{A}_2 \mathbf{Q}_1 &= \mathbf{0}, \\ \mathbf{A}_1 \mathbf{Q}_1 &= \mathbf{I}, \end{aligned}$$

where the three 2×2 dimensionless real matrices \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 are defined by

$$(3.12) \quad \begin{aligned} \mathbf{A}_1 &= (\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_2 &= (\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) + (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_3 &= (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}). \end{aligned}$$

The general solution to Eq. (3.10) can be written in the form

$$(3.13) \quad \mathbf{Q}_j = \mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2, \quad j = 1, 2, \dots, \infty,$$

where

$$(3.14) \quad \mathbf{\Lambda}_{1,2} = \frac{1}{2} \left[-\mathbf{A}_1^{-1} \mathbf{A}_2 \pm \sqrt{(\mathbf{A}_1^{-1} \mathbf{A}_2)^2 - 4\mathbf{A}_1^{-1} \mathbf{A}_3} \right],$$

and the two 2×2 constant matrices \mathbf{P}_1 and \mathbf{P}_2 can be uniquely determined from Eq. (3.11) as

$$(3.15) \quad \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_1^2 & \mathbf{\Lambda}_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_1^{-1} \\ -\mathbf{A}_1^{-1} \mathbf{A}_2 \mathbf{A}_1^{-1} \end{bmatrix}.$$

It is seen from Eqs. (3.10)–(3.12) that all of the 2×2 constant matrices \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ are real-valued and dimensionless and are in fact independent of the nature of the singularity at $z = id$. Now, the analytic vector functions defined in all of the four phases can be determined explicitly as

$$(3.16) \quad \mathbf{f}_1(z) = \frac{4}{\pi} \sum_{j=1}^{+\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2) (\hat{\mathbf{b}} - i\mathbf{C}_4^{-1}\hat{\mathbf{f}}) \ln[z - i(d + 2h(j-1))],$$

$\text{Im}\{z\} \leq 0;$

$$\begin{aligned}
(3.17) \quad \mathbf{f}_2(z) &= \frac{2}{\pi}(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} - \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z - \mathbf{i}(d + 2h(j-1))] \\
&\quad + \frac{2}{\pi}(\mathbf{\Gamma}_1 - \mathbf{I}) \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \\
&\quad \times \ln[z + \mathbf{i}(d + 2h(j-1))], \quad 0 \leq \text{Im}\{z\} \leq h;
\end{aligned}$$

$$\begin{aligned}
(3.18) \quad \mathbf{f}_3(z) &= \frac{1}{\pi}(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} - \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z - \mathbf{i}(d + 2h(j-1))] \\
&\quad + \frac{1}{\pi}(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} - \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z - \mathbf{i}(d + 2jh)] \\
&\quad + \frac{1}{\pi}(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z + \mathbf{i}(d + 2h(j-1))] \\
&\quad + \frac{1}{\pi}(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z + \mathbf{i}(d + 2h(j-2))], \\
&\hspace{25em} h \leq \text{Im}\{z\} \leq 2h;
\end{aligned}$$

$$\begin{aligned}
(3.19) \quad \mathbf{f}_4(z) &= \frac{1}{2\pi}(\hat{\mathbf{b}} - \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln(z - \mathbf{id}) \\
&\quad + \frac{1}{2\pi}(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z + \mathbf{i}(d + 2h(j-1))] \\
&\quad + \frac{1}{2\pi}[(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) + (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I})] \\
&\quad \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1} \hat{\mathbf{f}}) \ln[z + \mathbf{i}(d + 2h(j-2))]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\pi}(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
& \times \sum_{j=1}^{\infty} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2)(\hat{\mathbf{b}} + \mathbf{iC}_4^{-1}\hat{\mathbf{f}}) \ln[z + \mathbf{i}(d + 2h(j - 3))], \\
& \text{Im}\{z\} \geq 2h.
\end{aligned}$$

Using the extended version of the Peach-Koehler formula [17, 18], the image force acting on the piezoelectric screw dislocation is finally derived as

$$(3.20) \quad F_1 = 0, \quad F_2 = \frac{C_0}{4\pi h}(\hat{\mathbf{b}}^T \mathbf{C}_4 \mathbf{M} \hat{\mathbf{b}} + \hat{\mathbf{f}}^T \mathbf{M} \mathbf{C}_4^{-1} \hat{\mathbf{f}}),$$

where F_1 and F_2 are, respectively, the force components along the x_1 and x_2 directions and the 2×2 dimensionless real matrix \mathbf{M} is defined by

$$(3.21) \quad \mathbf{M} = \mathbf{B}_1 \mathbf{G}(\tilde{d}) + \mathbf{B}_2 \mathbf{G}(\tilde{d} - 1) + \mathbf{B}_3 \mathbf{G}(\tilde{d} - 2), \quad \tilde{d} = \frac{d}{h} > 2,$$

where the three 2×2 dimensionless real matrices \mathbf{B}_1 , \mathbf{B}_2 and \mathbf{B}_3 are defined by

$$\begin{aligned}
(3.22) \quad \mathbf{B}_1 &= (\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\
\mathbf{B}_2 &= (\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) + (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\
\mathbf{B}_3 &= (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}),
\end{aligned}$$

and the 2×2 dimensionless real matrix function $\mathbf{G}(x)$ is given by

$$(3.23) \quad \mathbf{G}(x) = \sum_{j=1}^{\infty} \frac{1}{x + j - 1} (\mathbf{\Lambda}_1^j \mathbf{P}_1 + \mathbf{\Lambda}_2^j \mathbf{P}_2), \quad x > 0.$$

It is clear that the image force in Eq. (3.20) can be completely determined once the single 2×2 real matrix function $\mathbf{G}(x)$ is known. It is rigorously verified that: (i) when $\mathbf{C}_3 = \mathbf{C}_4$ (i.e., the electroelastic constants for S_3 and S_4 are the same), the present solution simply reduces to that by WANG and SCHIAVONE [10] for a three-phase piezoelectric laminate; (ii) when all four phases are elastic dielectric with the piezoelectric constants being zero for all phases and when $\hat{\mathbf{b}} = [b_3 \ 0]^T$, $\hat{\mathbf{f}} = \mathbf{0}$, the image force in Eq. (3.20) reduces to Eq. (3.24) by WANG and SCHIAVONE [9]. Thus the correctness of the present solution is partially verified.

We emphasize here that our method remains valid for a singularity of arbitrary type (e.g., an Eshelby inclusion of arbitrary shape). If the principal (or singular) part of $\mathbf{f}_4(z)$ is $\mathbf{f}_0(z)$, which is simply the analytic vector function for the singularity in an infinite homogeneous piezoelectric medium, then

$$(3.24) \quad \mathbf{\Omega}(z) = 8 \sum_{j=1}^{\infty} \mathbf{Q}_j \mathbf{f}_0[z - 2\mathbf{i}h(j - 1)],$$

where \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ are again determined by Eq. (3.13). Substituting the above into Eqs. (3.5)–(3.8) will yield the explicit expressions for the analytic vector functions defined in each of the four phases.

4. An illustrative example

In this section, a specific example is presented to demonstrate the theory developed in the previous section. In this example, phases 1 and 3 are composed of BaTiO₃ whose electroelastic constants are given by

$$(4.1) \quad C_{44} = 4.4 \times 10^{10} \text{ N/m}^2, \quad e_{15} = 11.4 \text{ C/m}^2, \quad \epsilon_{11} = 9.8722 \times 10^{-9} \text{ F/m},$$

and phases 2 and 4 are PZT-5 with electroelastic constants given by

$$(4.2) \quad C_{44} = 2.11 \times 10^{10} \text{ N/m}^2, \quad e_{15} = 12.3 \text{ C/m}^2, \quad \epsilon_{11} = 8.1103 \times 10^{-9} \text{ F/m}.$$

In addition, we choose $C_0 = 2.11 \times 10^{10} \text{ N/m}^2$ and $\epsilon_0 = 8.1103 \times 10^{-9} \text{ F/m}$, the elastic and dielectric constants for PZT-5.

After normalization, we obtain

$$(4.3) \quad \mathbf{C}_1 = \mathbf{C}_3 = \begin{bmatrix} 2.0853 & 0.8715 \\ 0.8715 & -1.2172 \end{bmatrix}, \quad \mathbf{C}_2 = \mathbf{C}_4 = \begin{bmatrix} 1 & 0.9403 \\ 0.9403 & -1 \end{bmatrix}.$$

We calculate:

$$(4.4) \quad \begin{aligned} \mathbf{\Lambda}_1 &= \begin{bmatrix} -0.0803 & 0.1000 \\ -0.3990 & 0.2377 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} 0.1587 & -0.1302 \\ 0.5192 & -0.2551 \end{bmatrix}, \\ \mathbf{P}_1 &= -\mathbf{P}_2 = \begin{bmatrix} 0.5687 & -0.2489 \\ 0.9928 & -0.2225 \end{bmatrix}. \end{aligned}$$

The solution thus obtained is convergent since the eigenvalues of the two roots $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ in Eq. (4.4) are strictly inside the unit circle. Consequently, we can find the 2×2 real matrices \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ given by Eq. (3.13). It is verified numerically that the recurrence relations in Eqs. (3.10) and (3.11) are indeed satisfied. The variation of the 2×2 matrix function $\mathbf{G}(x)$ is illustrated in Fig. 1. The variations of the two 2×2 matrices $\mathbf{U} = \mathbf{C}_4 \mathbf{M}$ and $\mathbf{V} = \mathbf{M} \mathbf{C}_4^{-1}$, which are fundamental to the image force expression in Eq. (3.20), as functions of $\tilde{d} - 2$ are illustrated in Figs. 2 and 3. It is sufficient to truncate the series in Eq. (3.23) at $j = 10$ to arrive at highly accurate results as shown in Figs. 1–3. It is observed from Figs. 2 and 3 that:

- (i) Each of the two matrices \mathbf{U} and \mathbf{V} is symmetric and thus the quadratic forms in Eq. (3.20) are standard;
- (ii) Both \mathbf{U} and \mathbf{V} are not positive definite or negative definite;

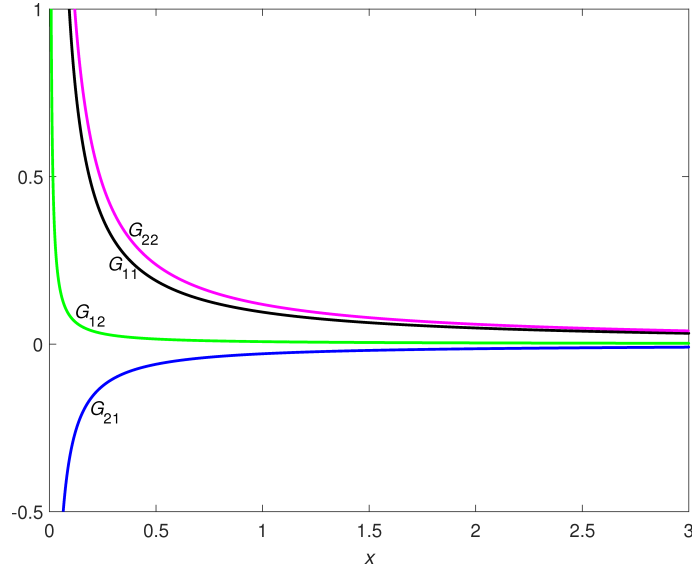


FIG. 1. Variation of the 2×2 matrix function $\mathbf{G}(x) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$.

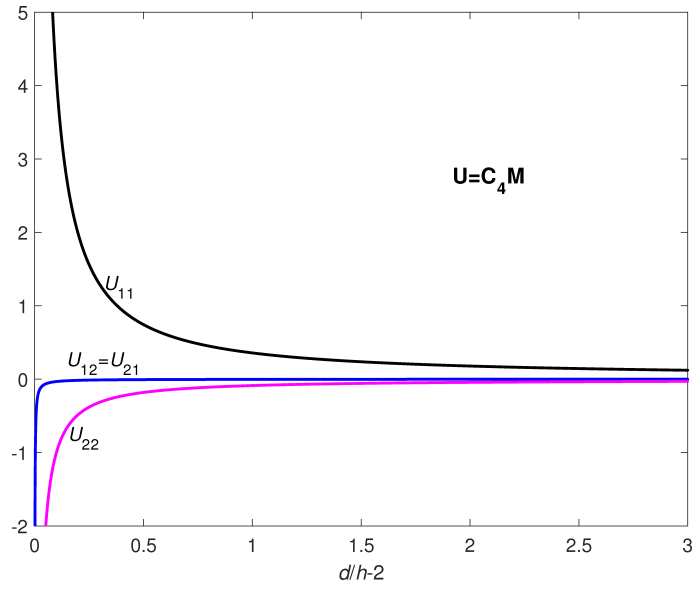


FIG. 2. Variation of the 2×2 matrix $\mathbf{C}_4\mathbf{M} = \mathbf{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ as a function of $\tilde{d} - 2$.

(iii) $U_{12} = U_{21} < 0$ and $V_{12} = V_{21} > 0$ are nonzero due to the piezoelectric effect;

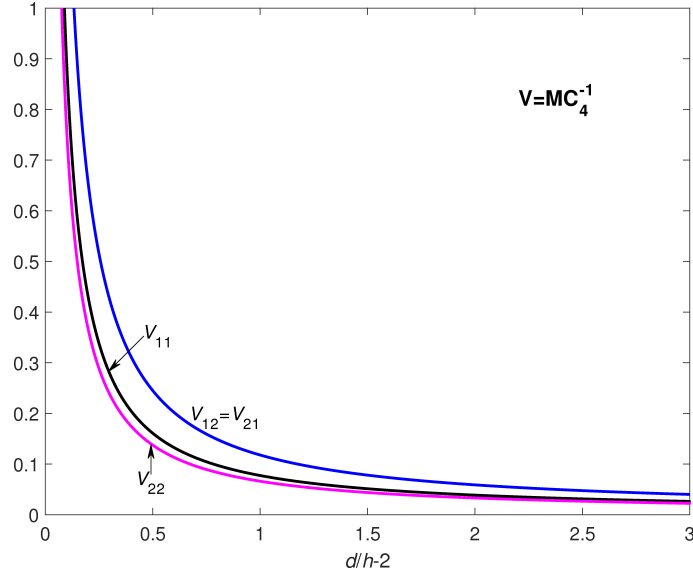


FIG. 3. Variation of the 2×2 matrix $\mathbf{MC}_4^{-1} = \mathbf{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$ as a function of $\tilde{d} - 2$.

(iv) As $\tilde{d} - 2 \rightarrow 0$,

$$\mathbf{U} \cong \frac{1}{\tilde{d} - 2} \mathbf{C}_4 (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})^{-1},$$

$$\mathbf{V} \cong \frac{1}{\tilde{d} - 2} (\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})^{-1} \mathbf{C}_4^{-1},$$

which are independent of the properties of phases 1 and 2;

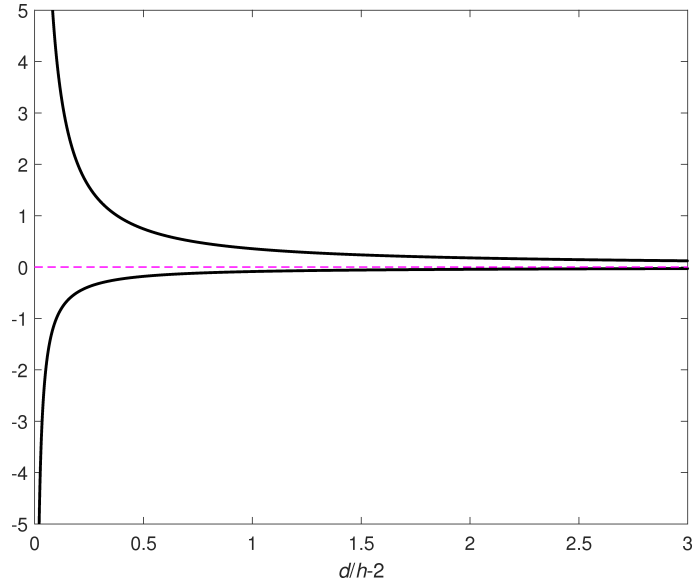
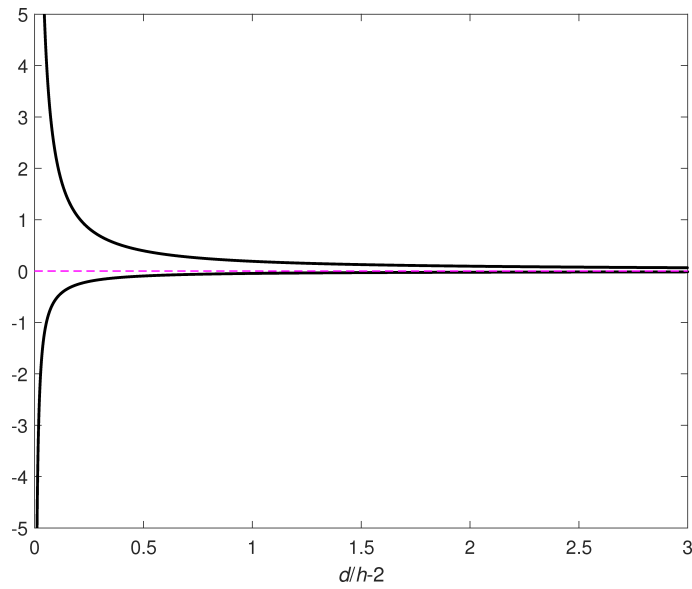
(v) As $\tilde{d} \rightarrow \infty$,

$$\mathbf{U} \cong \frac{1}{\tilde{d}} \mathbf{C}_4 (\mathbf{\Gamma}_3 \mathbf{\Gamma}_2 \mathbf{\Gamma}_1 - \mathbf{I})(\mathbf{\Gamma}_3 \mathbf{\Gamma}_2 \mathbf{\Gamma}_1 + \mathbf{I})^{-1},$$

$$\mathbf{V} \cong \frac{1}{\tilde{d}} (\mathbf{\Gamma}_3 \mathbf{\Gamma}_2 \mathbf{\Gamma}_1 - \mathbf{I})(\mathbf{\Gamma}_3 \mathbf{\Gamma}_2 \mathbf{\Gamma}_1 + \mathbf{I})^{-1} \mathbf{C}_4^{-1},$$

which are independent of the properties of the two intermediate layers.

Observations (iv) and (v) are in agreement with the results for a piezoelectric bi-material [19, 20]. Observation (ii) can be more clearly seen from the distributions of the two eigenvalues of \mathbf{U} in Fig. 4 and those of \mathbf{V} in Fig. 5. In Figs. 4 and 5, one eigenvalue is positive, whilst the other one is negative. This fact suggests that it is always possible to make any position in S_4 an equilibrium position (i.e., $F_2 \equiv 0$) using a judicious choice of the two ratios $\alpha = \hat{b}_2/\hat{b}_1$ and $\beta = \hat{f}_2/\hat{f}_1$ where $\hat{\mathbf{b}} = [\hat{b}_1 \ \hat{b}_2]^T$ and $\hat{\mathbf{f}} = [\hat{f}_1 \ \hat{f}_2]^T$. Figures 6 and 7 illustrate the two critical

FIG. 4. Distributions of the two eigenvalues of \mathbf{U} as functions of $\tilde{d} - 2$.FIG. 5. Distributions of the two eigenvalues of \mathbf{V} as functions of $\tilde{d} - 2$.

values of α and β , as functions of $\tilde{d} - 2$, at which $F_2 \equiv 0$. It is observed from these two figures that the two critical values of either α or β are almost constant for different values of $\tilde{d} - 2$. More specifically,

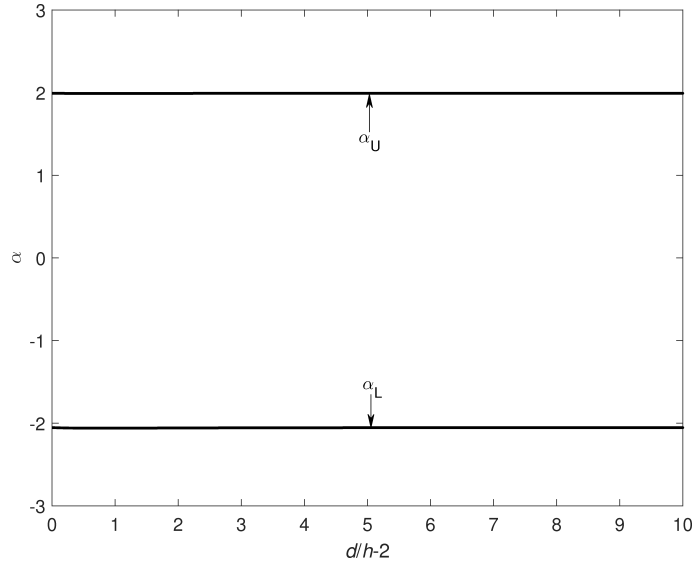


FIG. 6. The two critical values α_L and α_U of α , at which $F_2 \equiv 0$, as functions of $\tilde{d} - 2$.

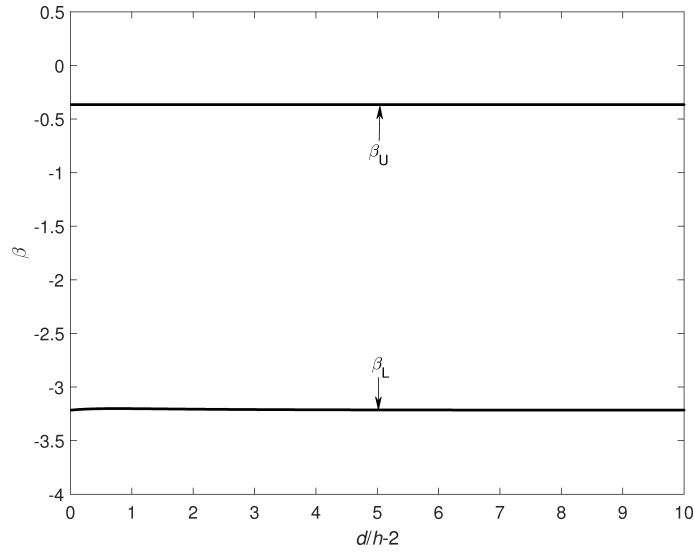


FIG. 7. The two critical values β_L and β_U of β , at which $F_2 \equiv 0$, as functions of $\tilde{d} - 2$.

$$\begin{aligned}
 (4.5) \quad & \min\{\alpha_L\} = -2.0605, \quad \max\{\alpha_L\} = -2.0535, \\
 & \min\{\alpha_U\} = 1.9902, \quad \max\{\alpha_U\} = 1.9910, \\
 & \min\{\beta_L\} = -3.2163, \quad \{\beta_L\} = -3.2012, \\
 & \min\{\beta_U\} = -0.3658, \quad \max\{\beta_U\} = -0.3657,
 \end{aligned}
 \quad \text{for } 0 < \tilde{d} - 2 < \infty.$$

5. Discussion of an N -phase piezoelectric laminate

Consider an entire z -plane composed of N (≥ 5) piezoelectric phases:

$$\begin{aligned} S_1 : \operatorname{Im}\{z\} \leq 0; \quad S_j : h(j-2) \leq \operatorname{Im}\{z\} \leq h(j-1), \quad j = 2, 3, \dots, N-1; \\ S_N : \operatorname{Im}\{z\} \geq h(N-2), \end{aligned}$$

where h (> 0) is the common thickness for each one of the $N-2$ intermediate layers. Each piezoelectric phase is hexagonal with its poling direction along the x_3 -axis. All of the N phases are perfectly bonded across the $N-1$ planar interfaces. The piezoelectric screw dislocation is located at $z = id$, $d > h(N-2)$ in S_N . The subscript j is used to identify the associated quantities in the phase S_j .

The analytic vector function $\mathbf{f}_1(z)$ defined in the lower semi-infinite piezoelectric medium continues to take the general form in Eq. (3.5). By satisfying the continuity conditions across the $N-1$ perfect interfaces and applying analytic continuation, the analytic vector function $\mathbf{f}_N(z)$ defined in the upper semi-infinite piezoelectric medium can be finally derived as

$$(5.1) \quad 2^{N-1} \mathbf{f}_N(z) = \sum_{j=1}^{N-1} \mathbf{A}_j \boldsymbol{\Omega}[z - 2ih(j-1)] + \sum_{j=1}^{N-1} \mathbf{B}_j \bar{\boldsymbol{\Omega}}[z - 2ih(j-1)],$$

$$\operatorname{Im}\{z\} \geq h(N-2),$$

where \mathbf{A}_j , \mathbf{B}_j , $j = 1, 2, \dots, N-1$ are 2×2 dimensionless real matrices which can be completely determined by $\boldsymbol{\Gamma}_j = \mathbf{C}_{j+1}^{-1} \mathbf{C}_j$, $j = 1, 2, \dots, N-1$. It is seen that \mathbf{A}_j and \mathbf{B}_j are fundamental to our solution.

An examination of Eq. (5.1) suggests that $\boldsymbol{\Omega}(z)$ should take the following specified form

$$(5.2) \quad \boldsymbol{\Omega}(z) = \frac{2^{N-2}}{\pi} \sum_{j=1}^{\infty} \mathbf{Q}_j (\hat{\mathbf{b}} - i\mathbf{C}_N^{-1} \hat{\mathbf{f}}) \ln[z - i(d + 2h(j-1))],$$

where $\hat{\mathbf{b}}$ and $\hat{\mathbf{f}}$ have been defined in Eq. (3.4), and \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ are 2×2 constant matrices to be determined.

Substitution of the above expression into Eq. (5.1) yields the following linear recurrence relations in matrix form:

$$(5.3) \quad \sum_{j=1}^{N-1} \mathbf{A}_{N-j} \mathbf{Q}_{j+m} = \mathbf{0}, \quad m = 0, 1, 2, \dots, \infty,$$

$$(5.4) \quad \sum_{j=m+1}^{N-1} \mathbf{A}_{N-j} \mathbf{Q}_{j-m} = \mathbf{0}, \quad m = 1, 2, \dots, N-3,$$

$$\mathbf{A}_1 \mathbf{Q}_1 = \mathbf{I},$$

in order to ensure that $\mathbf{f}_N(z)$ exhibits the logarithmic singularity

$$\mathbf{f}_N(z) \cong \frac{1}{2\pi}(\hat{\mathbf{b}} - \mathbf{i}\mathbf{C}_N^{-1}\hat{\mathbf{f}})\ln(z - id) + O(1) \quad \text{as } z \rightarrow id$$

and remains regular at the points

$$z = i(d + 2jh), \quad j = 1, 2, \dots, \infty.$$

The general solution to the linear recurrence relationship in matrix form in Eq. (5.3) can be derived as

$$(5.5) \quad \mathbf{Q}_j = \sum_{m=1}^{N-2} \mathbf{\Lambda}_m^j \mathbf{P}_m, \quad j = 1, 2, \dots, \infty,$$

where $\mathbf{\Lambda}_m$ are the $N - 2$ roots of the following algebraic equation of order $N - 2$ of matrix form in $\mathbf{\Lambda}$

$$(5.6) \quad \mathbf{A}_1 \mathbf{\Lambda}^{N-2} + \mathbf{A}_2 \mathbf{\Lambda}^{N-3} + \dots + \mathbf{A}_{N-2} \mathbf{\Lambda} + \mathbf{A}_{N-1} = \mathbf{0},$$

and the $N - 2$ coefficients \mathbf{P}_m can be uniquely determined from the set of the linear algebraic equations in Eq. (5.4).

It is difficult to find an analytical solution of the algebraic equation in matrix form in Eq. (5.6) for $N \geq 5$. In fact, even an analytical solution to the cubic equation of matrix form for $N = 5$ in Eq. (5.6) is unavailable. However, all of the $N - 2$ roots of Eq. (5.6) can be obtained via iteration. Another more direct solution method is based on the fact that the constant matrices \mathbf{Q}_j , $j = 1, 2, \dots, \infty$, all of which are real and dimensionless and independent of the nature of the singularity at $z = id$, can be easily determined from Eqs. (5.3) and (5.4) in a recursive manner once the $N - 1$ matrices \mathbf{A}_j , $j = 1, 2, \dots, N - 1$ are given.

The image force acting on the screw dislocation can also be derived as

$$(5.7) \quad F_1 = 0, \quad F_2 = \frac{C_0}{4\pi h}(\hat{\mathbf{b}}^T \mathbf{C}_N \mathbf{M} \hat{\mathbf{b}} + \hat{\mathbf{f}}^T \mathbf{M} \mathbf{C}_N^{-1} \hat{\mathbf{f}}),$$

where the 2×2 dimensionless real matrix \mathbf{M} is defined as follows

$$(5.8) \quad \mathbf{M} = \sum_{j=1}^{N-1} \mathbf{B}_j \mathbf{G}[\tilde{d} - (j - 1)], \quad \tilde{d} = \frac{d}{h} > N - 2,$$

where the 2×2 dimensionless real matrix function $\mathbf{G}(x)$ is given by

$$(5.9) \quad \mathbf{G}(x) = \sum_{j=1}^{\infty} \frac{\mathbf{Q}_j}{x + j - 1}, \quad x > 0,$$

which is, in fact, consistent with Eq. (3.23). The image force in Eq. (5.7) can be arrived at once the single real matrix function $\mathbf{G}(x)$ is known.

Our method remains valid for a singularity of arbitrary type. If the principal (or singular) part of $\mathbf{f}_N(z)$ is $\mathbf{f}_0(z)$, then

$$(5.10) \quad \Omega(z) = 2^{N-1} \sum_{j=1}^{\infty} \mathbf{Q}_j \mathbf{f}_0[z - 2ih(j-1)],$$

where \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ are determined from Eq. (5.5) or from the recursive solution of Eqs. (5.3) and (5.4).

Below, we present specific results for $N = 5, 6$.

For a five-phase piezoelectric laminate ($N = 5$), the explicit expressions of the matrices \mathbf{A}_j , \mathbf{B}_j , $j = 1, 2, 3, 4$ are

$$(5.11) \quad \begin{aligned} \mathbf{A}_1 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_2 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_3 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\ \mathbf{A}_4 &= (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \end{aligned}$$

$$(5.12) \quad \begin{aligned} \mathbf{B}_1 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\ \mathbf{B}_2 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\ \mathbf{B}_3 &= (\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{B}_4 &= (\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}). \end{aligned}$$

Furthermore, phases 1, 3 and 5 are taken as BaTiO₃ with its electroelastic constants given by Eq. (4.1), phases 2 and 4 are taken as PZT-5 with its electroelastic constants given by Eq. (4.2), and $C_0 = 2.11 \times 10^{10}$ N/m², and $\epsilon_0 = 8.1103 \times 10^{-9}$ F/m. We calculate:

(5.13)

$$\begin{aligned}\mathbf{\Lambda}_1 &= \begin{bmatrix} 0.3318 & -0.0558 \\ 0.2225 & 0.1545 \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \begin{bmatrix} -0.3769 & 0.2343 \\ -0.9346 & 0.3679 \end{bmatrix}, \quad \mathbf{\Lambda}_3 = \begin{bmatrix} 0.1627 & -0.2237 \\ 0.8924 & -0.5485 \end{bmatrix}, \\ \mathbf{P}_1 &= \begin{bmatrix} 0.0435 & 0.0173 \\ -0.0691 & 0.0986 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0.0716 & -0.0550 \\ 0.2194 & -0.1033 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} -0.1151 & 0.0377 \\ -0.1504 & 0.0047 \end{bmatrix}.\end{aligned}$$

The three roots $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_3$ are determined in the following manner: First, the root $\mathbf{\Lambda}_1$ is iteratively determined from the cubic equation in Eq. (5.6) with $N = 5$. Second, the cubic equation in Eq. (5.6) can be reduced to a quadratic equation. Third, the other two roots $\mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_3$ can be determined by rigorously solving the quadratic equation. The resulting solution is convergent since the eigenvalues of the three roots $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_3$ in Eq. (5.13) lie strictly inside the unit circle. The matrices \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ can be determined from Eq. (5.5) and they indeed satisfy the recurrence relations in Eqs. (5.3) and (5.4). The variation of the 2×2 matrix function $\mathbf{G}(x)$ obtained from Eqs. (5.5) and (5.9) is illustrated in Fig. 8. The variations of the two 2×2 matrices $\mathbf{U} = \mathbf{C}_5 \mathbf{M}$ and $\mathbf{V} = \mathbf{M} \mathbf{C}_5^{-1}$ as functions of $\tilde{d} - 3$ are illustrated in Figs. 9 and 10. It is also sufficient to truncate the series in Eq. (5.9) at $j = 10$ to arrive at highly accurate results in Figs. 8-10. In this example, the two matrices \mathbf{U} and \mathbf{V} are also symmetric.

For a six-phase piezoelectric laminate ($N = 6$), the explicit expressions of the matrices \mathbf{A}_j , \mathbf{B}_j , $j = 1, 2, 3, 4, 5$ are given by

$$\begin{aligned}(5.14) \quad \mathbf{A}_1 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_2 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_3 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\ \mathbf{A}_4 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\ &\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\ \mathbf{A}_5 &= (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}),\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_1 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\
\mathbf{B}_2 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\
\mathbf{B}_3 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}), \\
\mathbf{B}_4 &= (\mathbf{\Gamma}_5 + \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 - \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 - \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}) \\
&\quad + (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 - \mathbf{I})(\mathbf{\Gamma}_3 - \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}), \\
\mathbf{B}_5 &= (\mathbf{\Gamma}_5 - \mathbf{I})(\mathbf{\Gamma}_4 + \mathbf{I})(\mathbf{\Gamma}_3 + \mathbf{I})(\mathbf{\Gamma}_2 + \mathbf{I})(\mathbf{\Gamma}_1 + \mathbf{I}).
\end{aligned}
\tag{5.15}$$

In addition, phases 1, 3 and 5 are taken as PZT-5 with its electroelastic constants given by Eq. (4.2), phases 2, 4 and 6 are taken as BaTiO₃ with its

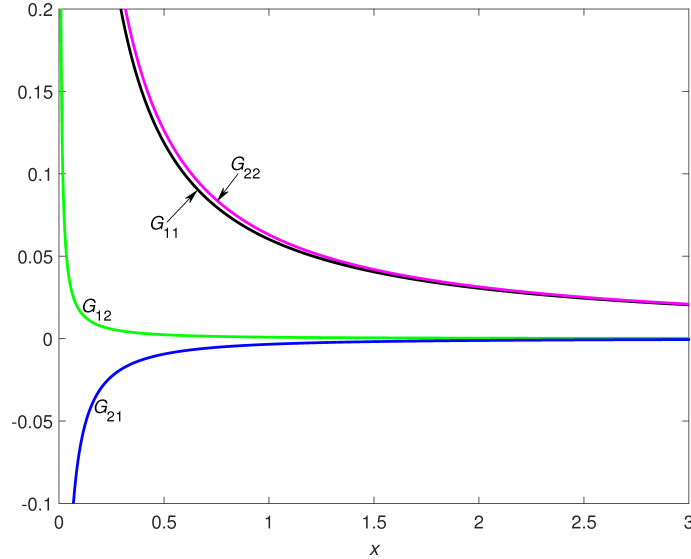


FIG. 8. Variation of the 2×2 matrix function $\mathbf{G}(x) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ for a five-phase piezoelectric laminate.

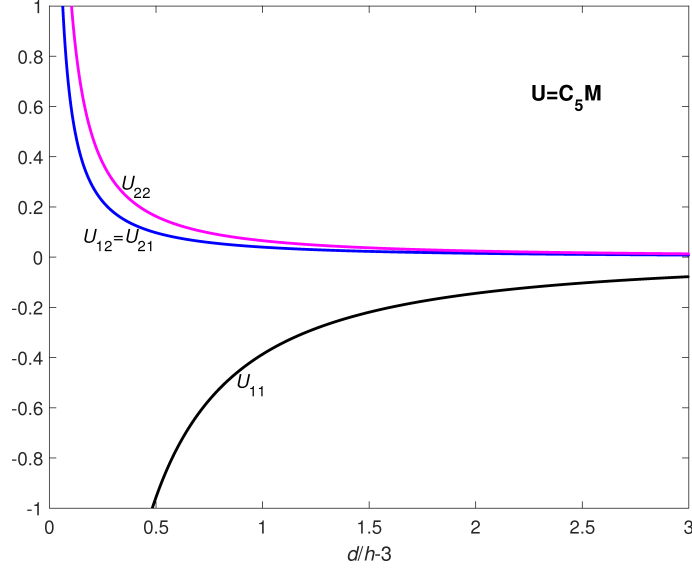


FIG. 9. Variation of the 2×2 matrix $\mathbf{C}_5 \mathbf{M} = \mathbf{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ as a function of $\tilde{d} - 3$ for a five-phase piezoelectric laminate.

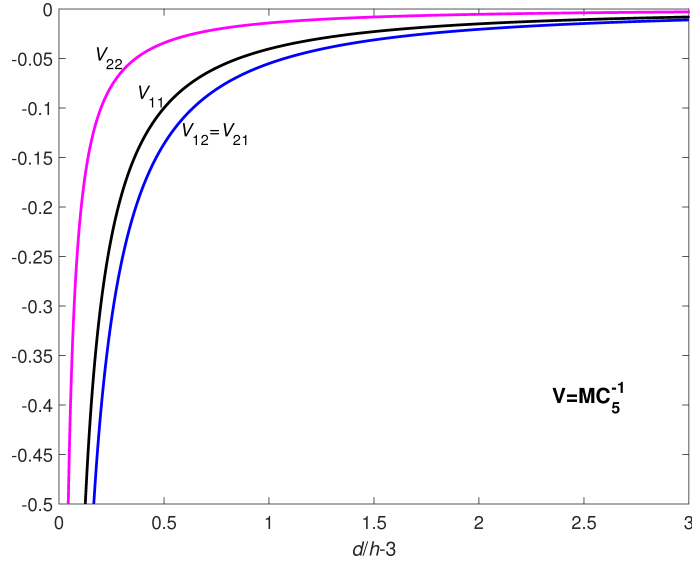


FIG. 10. Variation of the 2×2 matrix $\mathbf{M} \mathbf{C}_5^{-1} = \mathbf{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$ as a function of $\tilde{d} - 3$ for a five-phase piezoelectric laminate.

electroelastic constants given by Eq. (4.1), and $C_0 = 2.11 \times 10^{10}$ N/m², and $\epsilon_0 = 8.1103 \times 10^{-9}$ F/m. We calculate:

$$\begin{aligned}
 (5.16) \quad \mathbf{\Lambda}_1 &= \begin{bmatrix} 0.0919 & 0.1338 \\ -0.5337 & 0.5173 \end{bmatrix}, & \mathbf{\Lambda}_2 &= \begin{bmatrix} 0.1059 & -0.2790 \\ 1.1131 & -0.7811 \end{bmatrix}, \\
 \mathbf{\Lambda}_3 &= \begin{bmatrix} 0.5903 & -0.2316 \\ 0.9237 & -0.1459 \end{bmatrix}, & \mathbf{\Lambda}_4 &= \begin{bmatrix} -0.6313 & 0.3165 \\ -1.2626 & 0.3749 \end{bmatrix}, \\
 \mathbf{P}_1 &= \begin{bmatrix} 0.0365 & -0.0111 \\ 0.0443 & 0.0012 \end{bmatrix}, & \mathbf{P}_2 &= \begin{bmatrix} -0.0346 & 0.0091 \\ -0.0364 & -0.0057 \end{bmatrix}, \\
 \mathbf{P}_3 &= \begin{bmatrix} -0.0102 & 0.0153 \\ -0.0611 & 0.0385 \end{bmatrix}, & \mathbf{P}_4 &= \begin{bmatrix} 0.0082 & -0.0133 \\ 0.0531 & -0.0341 \end{bmatrix}.
 \end{aligned}$$

Thus, the matrices \mathbf{Q}_j , $j = 1, 2, \dots, \infty$ can be determined from Eq. (5.5) and they indeed satisfy the recurrence relations in Eqs. (5.3) and (5.4). The resulting solution is convergent because the eigenvalues of the four roots $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$, $\mathbf{\Lambda}_3$ and $\mathbf{\Lambda}_4$ in Eq. (5.16) lie strictly inside the unit circle.

6. Conclusions

We first propose a new version of the complex variable formulation of the interaction problem associated with a screw dislocation in a layered piezoelectric laminate in which the real symmetric matrix \mathbf{C} defined in Eq. (2.3) becomes dimensionless. Using this complex variable formulation and the technique of analytic continuation, the original boundary value problem for the four-phase piezoelectric laminate is reduced to the determination of the single analytic vector function $\mathbf{\Omega}(z)$. After rigorously solving the linear recurrence relations in matrix form in Eqs. (3.10) and (3.11), an elementary expression for $\mathbf{\Omega}(z)$ is obtained in Eq. (3.9). Explicit expressions of the four analytic vector functions $\mathbf{f}_1(z)$, $\mathbf{f}_2(z)$, $\mathbf{f}_3(z)$ and $\mathbf{f}_4(z)$ characterizing the electroelastic field in the four-phase piezoelectric laminate are presented in Eqs. (3.16)–(3.19). By using the extended version of the Peach–Koehler formula, the image force acting on the screw dislocation is given by Eq. (3.20) containing the single real matrix function $\mathbf{G}(x)$. The image force in Eq. (3.20) becomes relatively simple to interpret since the 2×2 matrices \mathbf{C}_4 and \mathbf{M} are dimensionless, whilst the 2D vectors $\hat{\mathbf{b}}$ and $\hat{\mathbf{f}}$ have the dimension of length. An example is presented to illustrate the proposed theory. Finally, the more general situation of a screw dislocation in an N -phase piezoelectric laminate is discussed and numerical results are presented for the cases $N = 5, 6$.

The present solutions can be conveniently employed as Green's functions to further study crack problems in piezoelectric laminates (the crack problem in a homogeneous piezoelectric plane can be found in [21, 22]).

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