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On Cherkaev–Lurie–Milton theorem in the plane problems of linear elasticity

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The paper delivers a full justification of the Cherkaev–Lurie–Milton theorem in application to the elasticity problem of in-plane loaded plates, 2D periodic elastic composites, elasticity of thin plates subjected to transverse loads as well as in-plane periodic thin plates in bending. The theorem is treated as natural extension of Michell's result on 2D elasticity and the Gauss–Bonnet formula applied to the deflection surface of a thin plate subject to bending.

Key words: Cherkaev–Lurie–Milton theorem (CLM theorem), effective stiffnesses of thin plates, theory of composites.

1. Introduction

The essential feature of the linear theory of structures constructed of Bernoulli–Euler or Timoshenko bars is the existence of a subclass of systems of bars, called *statically determinate structures*, in which the set of statically admissible internal force fields (i.e. axial forces, bending and torsional moments) is a singleton. Consequently, if the structure is not subject to a static load, then all the internal forces vanish. Moreover, the diagrams of the internal forces are independent of the distribution of stiffnesses along the axes of the bars.

Among the 2D linear elasticity problems only one: the pure torsion of an annular plate is statically determinate: the shear stress can be computed directly from one equilibrium equation. This shear stress is obviously independent of the elastic moduli. Only in this simple problem the set of statically admissible stresses is a singleton. Despite this fact there is well known a broad class of 2D static problems of elasticity of materially homogeneous isotropic plates (loaded in the plane), discovered by Michell [1], in which *the stress fields do not depend on the elastic moduli.* To make a stress field independent of the elastic moduli the contours of an in-plane loaded, possibly multi-connected, plate should be free of kinematic conditions, while the traction load, apart from three required self-equilibrium conditions, should give zero resultants along the contours of each opening. In case of 2D simply connected domains the stress is independent of the elastic moduli, if the whole contour is subject to (self-equilibrated) tractions. A very clear proof of the Michell theorem can be found in MUSKHELISHVILI [2].

Similar theorems concern isotropic and homogeneous thin plates in bending, where the role of the stress fields is played by the measures of changes of curvature, determined by the deflection function w. If Ω is the middle plane of the plate, its bending stiffness is denoted by D and ν is Poisson's ratio, the expression

$$\Pi(w) = \frac{1}{2}\int\limits_{\Omega} D\Big[(\Delta w)^2 - \underline{2(1-\nu)\det(\nabla\nabla w)}\Big]\,dx \tag{1.1}$$

represents the elastic energy due to bending within the linear theory of Kirchhoff. Let the plate be subject to the distributed transverse load of intensity q. Then the plate deflection w is the minimizer of the problem

$$\min_{v\in V(\Omega)}\Big\{\Pi(v) - \int\limits_{\Omega} qv\,dx\Big\}, \tag{1.2}$$

$V(\Omega)$ being the linear affine set of kinematically admissible deflections. Localization of (1.2) leads to the governing equation:

$$D\Delta^2 w = q. \tag{1.3}$$

The Gauss curvature of the surface formed by the deformed plate is expressed by

$$K = \frac{\det(\nabla\nabla w)}{[1+\|\nabla w\|^2]^2}. \tag{1.4}$$

According to the Gauss–Bonnet theorem the integral of K over the surface is expressed, up to a constant, by the contour integral of the geodesic curvature, see SZABÓ [3]. In case of the gradients of $w(x)$ being small with respect to 1 the numerator of (1.4) represents the first order approximation of K, hence the term underlined in (1.1) can be reduced to the contour integral and should not affect the form of the local equation (1.3). This suggests (yet the proof is still needed) that the deflection of an isotropic and homogeneous plate, fully clamped along its contour, does not depend on the Poisson ratio.

The historically first derivation of the governing equation (1.3) is attributed to S. Germain (and, sometimes, to J. L. Lagrange who corrected mistakes in S. Germain's derivation). Yet let us stress that this original derivation started from the formulations (1.1), (1.2) with the underlined term in (1.1) being omitted. Knowing that an untruth can imply a truth, there is no contradiction here, yet the history of the thin plate theory could have been less quixotic. Eventually, the correct and complete linear theory of thin plates in bending has been constructed by G. Kirchhoff, see SZABÓ [3].

The two theorems known from 19th century:

– theorem by Michell [1] on the conditions assuring the stress state within an in-plane loaded isotropic and homogeneous plate being independent of the elastic moduli,

– the mentioned theorem concerning the conditions of the deflection of a thin isotropic plate in bending being independent of Poisson's ratio,

have been extended in the 1980's to the theory of composites, see the review paper by Lurie and Cherkaev [4]. It has occurred that the effective moduli of periodic composites are independent of certain uniform translation of flexibilities of the constituents. This stability result is known as the Cherkaev–Lurie–Milton theorem, presented in its final form in the papers by Cherkaev *et al.* [5] and Thorpe and Jasiuk [6], both published in the same volume of the Proceedings of the Royal Society of London. The aim of the present paper is to deliver the full documentation of this theorem in its four areas of application:

- static problems of in-plane loaded thin plates,
- homogenization of flexibilities of in-plane loaded planar composites,
- static problems of transversely loaded thin plates,
- homogenization of bending stiffnesses of transversely loaded thin plates.

The proofs of these theorems are scattered in the literature, hence it is thought appropriate to gather the material in the form of one publication.

The present paper does not deal with the regularity problems, hence the operations *inf* and *sup* do not appear.

We shall make use of the notation introduced in paper [7]. In particular, the Cartesian basis vectors will be denoted by: $\mathbf{e}_i$, $i = 1, 2$; $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where $\cdot$ is the scalar product in $\mathbb{R}^2$. The Euclidean norm of $\mathbf{p} \in \mathbb{R}^2$ is defined by $\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \mathbf{p}}$. The set of second rank symmetric tensors is denoted by E_s^2. The identity tensor in E_s^2 is $\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$; repetition of indices implies summation. The set of fourth rank tensors $\mathbf{C} = C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ satisfying the symmetry conditions $C_{ijkl} = C_{klij}$, $C_{ijkl} = C_{jikl}$ is denoted by E_s^4. The components of the identity tensor in E_s^4 read $(\mathbf{II})_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The product of $\mathbf{A}, \mathbf{B} \in E_s^4$ is defined by the rule: $(\mathbf{AB})_{ijkl} = A_{ijmn}B_{mnkl}$. The scalar product of $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in E_s^2$ is defined by: $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$. The Euclidean norm of $\boldsymbol{\sigma} \in E_s^2$ reads $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}$. Let us define

$$
\begin{aligned}
\operatorname{Tr} \boldsymbol{\sigma} &= \frac{1}{\sqrt{2}}(\sigma_{11} + \sigma_{22}), \\
\operatorname{dev} \boldsymbol{\sigma} &= \boldsymbol{\sigma} - \left(\frac{1}{\sqrt{2}} \operatorname{Tr} \boldsymbol{\sigma}\right)\mathbf{I}, \\
\det \boldsymbol{\sigma} &= \sigma_{11}\sigma_{22} - (\sigma_{12})^2.
\end{aligned}
\tag{1.5}
$$

The norm of the deviator equals

$$\| \operatorname{dev} \boldsymbol{\sigma} \| = \sqrt{\frac{1}{2}(\sigma_{11} - \sigma_{22})^2 + 2(\sigma_{12})^2}. \tag{1.6}$$

A comma implies differentiation with respect to a Cartesian coordinate: $(.)_{,i} = \partial(.)/\partial x_i$ or $(.)_{,i} = \partial(.)/\partial y_i$, depending on the context.

2. The 2D elasticity problem with natural boundary conditions. The stress-based approach

Assume that the 2D body occupies a multi-connected domain Ω whose external contour Γ_0 as well as the internal contours $\Gamma_1, \ldots, \Gamma_n$ are subject to tractions of intensity $\mathbf{T} = (T_1, T_2)$. The body forces are omitted. Let Γ be the sum of the contours Γ_i, $i = 0, \ldots, n$; each contour being parameterized by the natural parameter s_i, $\mathbf{n} = (n_1, n_2)$ is the unit vector outward normal to the contour Γ, cf. MUSKHELISHVILI [2]. The tractions are subject to the conditions

$$\int\limits_{\Gamma_i} T_1 \, ds_i = 0, \qquad \int\limits_{\Gamma_i} T_2 \, ds_i = 0, \qquad i = 0, \ldots, n, \tag{2.1}$$

$$\sum_{i=0}^{n} \int\limits_{\Gamma_i} (x_1 T_2 - x_2 T_1) \, ds_i = 0. \tag{2.2}$$

Thus, the load $\mathbf{T}$ is self-equilibrated, gives zero resultants on the each contour, but does not need to produce zero moments along the contours $\Gamma_1, \ldots, \Gamma_n$. The conditions (2.1), (2.2) are stronger than usual conditions of the tractions being self-equilibrated. The domain Ω is parameterized by the Cartesian orthogonal system x_1, x_2.

We say that the test stress field $\boldsymbol{\tau} = (\tau_{ij})$ in Ω, locally in E_s^2, satisfying

– the local equilibrium equations

$$\tau_{ij,j} = 0 \quad \text{in } \Omega, \tag{2.3}$$

– the natural boundary conditions

$$\tau_{ij} n_i = T_j \quad \text{in } \Gamma \tag{2.4}$$

is *statically admissible*. Such test stress fields form the linear affine set $\Sigma_T(\Omega)$.

Now, we shall determine the stress field $\boldsymbol{\sigma}$ which arises in the body loaded by the given tractions. Let $\mathbf{c}(x)$ be the tensor of elastic flexibilities and $\mathbf{c}^{-1}(x) = \mathbf{C}(x)$

be the tensor of elastic moduli, both of the class E_s^4 and both positive definite. The energy density corresponding to the test stress field $\boldsymbol{\tau}$ is given by

$$W(\boldsymbol{\tau}(x)) = \tfrac{1}{2}\boldsymbol{\tau}(x)\cdot(\mathbf{c}(x)\boldsymbol{\tau}(x)). \tag{2.5}$$

According to Castigliano's theorem, see Duvaut and Lions [8, Sec. 3.5], the stress field $\boldsymbol{\sigma}$ arising in the body is the minimizer of the variational problem

$$\min_{\boldsymbol{\tau}\in\Sigma_T(\Omega)} \Im(\boldsymbol{\tau}) = \Im(\boldsymbol{\sigma}), \tag{2.6}$$

where

$$\Im(\boldsymbol{\tau}) = \int_\Omega W(\boldsymbol{\tau}(x))\,dx \tag{2.7}$$

and the minimizer is unique, provided that the conditions (2.1)–(2.2) are fulfilled. We remember that the conditions (2.1) are here too strong, but their form is necessary in the sequel. If $\boldsymbol{\sigma}$ solves (2.6) then there exists a displacement field $\mathbf{u} = (u_1, u_2)$ such that

$$\mathbf{c}(x)\boldsymbol{\sigma}(x) = \boldsymbol{\varepsilon}(\mathbf{u}(x)) \quad \text{in } \Omega, \tag{2.8}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ is the strain tensor. The components of strain satisfy the following compatibility equation

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0. \tag{2.9}$$

Recovery of the displacements is possible by appropriate integration of strains.

To each stress field $\boldsymbol{\tau} \in \Sigma_T(\Omega)$ one can assign an Airy stress function F such that

$$\tau_{11} = F_{,22} \qquad \tau_{22} = F_{,11} \qquad \tau_{12} = -F_{,12} \tag{2.10}$$

locally within Ω, and

$$\frac{\partial}{\partial s}(F_{,2}) = T_1, \qquad \frac{\partial}{\partial s}(F_{,1}) = -T_2 \qquad \text{on } \Gamma, \tag{2.11}$$

see [2, Eqs. (3), Sec. 32, p. 111]. Integration of (2.11) introduces constants, independent for each opening, hence

$$F_{,1} = -\int_0^{s_i} T_2\,ds_i + C_i, \qquad F_{,2} = \int_0^{s_i} T_1\,ds_i + D_i, \qquad i = 0,\ldots,n, \tag{2.12}$$

where s_i is the natural parameter along the i-th contour.

Let

$$J_i = \int_{\Gamma_i} \nabla F \cdot \mathbf{T}\, ds_i \tag{2.13}$$

and

$$R_i^{(1)} = \int_0^{s_i} T_1\, ds_i, \qquad R_i^{(2)} = -\int_0^{s_i} T_2 ds_i. \tag{2.14}$$

Substitution of (2.14) into (2.7) makes it possible to rewrite (2.3) in the form

$$J_i = \int_{\Gamma_i} (R_2^{(i)} T_1 + R_1^{(i)} T_2)\, ds_i + C_i \int_{\Gamma_i} T_1\, ds_i + D_i \int_{\Gamma_i} T_2\, ds_i. \tag{2.15}$$

By virtue of the assumptions (2.1) the above integrals reduce to

$$J_i = \int_{\Gamma_i} (R_2^{(i)} T_1 + R_1^{(i)} T_2)\, ds_i, \tag{2.16}$$

which reveals that J_i, $i = 0, \ldots, n$, do not depend on the function F, hence J_i are determined only by the geometry of the domain Ω and the distribution of the traction load along the contours; thus, we shall write: $J_i = J_i(\mathbf{T})$.

Consider now the case of isotropy. The Hooke tensor is represented by Hill's formula

$$\mathbf{C} = 2k\,(x)\,\boldsymbol{\Lambda}_1 + 2\mu(x)\boldsymbol{\Lambda}_2, \tag{2.17}$$

where

$$(\boldsymbol{\Lambda}_1)_{ijkl} = \tfrac{1}{2}\delta_{ij}\delta_{kl}, \qquad (\boldsymbol{\Lambda}_2)_{ijkl} = (\mathbf{II})_{ijkl} - (\boldsymbol{\Lambda}_1)_{ijkl}. \tag{2.18}$$

The tensors $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ have the properties of projectors, see WALPOLE [9]

$$\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_1, \qquad \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2 = \mathbf{0}, \qquad \boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_2. \tag{2.19}$$

Thus, the decomposition (2.17) of Hooke tensors of isotropic materials comprises two tensors being mutually orthogonal. In 2D elasticity the bulk and shear moduli: (k, μ) are linked with the 2D Young modulus E and with the Poisson ratio ν by

$$k = \frac{E}{2(1-\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{2.20}$$

Tensor $\mathbf{C}$ is positive definite if $k > 0$ and $\mu > 0$. The tensor $\mathbf{c}$ inverse to $\mathbf{C}$ is represented by

$$\mathbf{c} = \frac{1}{2k(x)}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu(x)}\boldsymbol{\Lambda}_2. \tag{2.21}$$

Let us note, that for the test stress $\boldsymbol{\tau} \in E_s^2$ we have

$$\boldsymbol{\tau} \cdot (\boldsymbol{\Lambda}_1 \boldsymbol{\tau}) = (\operatorname{Tr} \boldsymbol{\tau})^2, \qquad \boldsymbol{\tau} \cdot (\boldsymbol{\Lambda}_2 \boldsymbol{\tau}) = \| \operatorname{dev} \boldsymbol{\tau} \|^2. \tag{2.22}$$

Thus, the elastic energy stored in an in-plane loaded plate is expressed by the formula

$$\Im(\boldsymbol{\tau}) = \frac{1}{2} \int\limits_{\Omega} \frac{1}{E} \Big[(\tau_{11} + \tau_{22})^2 - \underline{2(1+\nu)(\tau_{11}\tau_{22} - (\tau_{12})^2)} \Big] \, dx. \tag{2.23}$$

CHERKAEV, LURIE and MILTON [5] noted that the term underlined in (2.23) can be interpreted as a result of translation of the values of all the components of flexibilities of the in-plane loaded plate by a certain homogenous isotropic tensor. Indeed, let us define the *translation operator* using the projectors (2.18) by

$$\mathbf{L}_\lambda = \frac{1}{2\lambda} \mathbf{L}, \qquad \mathbf{L} = \boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2, \tag{2.24}$$

where λ is a constant; here $1/\lambda$ has the dimension of the flexibility of the plate loaded in plane. Due to (2.22)

$$\boldsymbol{\tau} \cdot (\mathbf{L}_\lambda \boldsymbol{\tau}) = \frac{1}{2\lambda} \big[(\operatorname{Tr} \boldsymbol{\tau})^2 - \|\operatorname{dev} \boldsymbol{\tau}\|^2 \big] \tag{2.25}$$

or

$$\boldsymbol{\tau} \cdot (\mathbf{L}_\lambda \boldsymbol{\tau}) = \frac{1}{\lambda} \det \boldsymbol{\tau}. \tag{2.26}$$

The formula (2.26) reveals the main idea of introducing the translator (2.24).

Note that each $\boldsymbol{\sigma} \in E_s^2$ can be decomposed into two mutually orthogonal components by the rule: $\boldsymbol{\sigma} = \boldsymbol{\Lambda}_1 \boldsymbol{\sigma} + \boldsymbol{\Lambda}_2 \boldsymbol{\sigma}$. By treating $\boldsymbol{\sigma}$ as a vector in $\mathbb{R}^3$ according to Bechterew's concept, see LEWIŃSKI [7], and by reflecting it with respect to the axis determined by $\boldsymbol{\Lambda}_1 \boldsymbol{\sigma}$, we obtain $\breve{\boldsymbol{\sigma}} = \boldsymbol{\Lambda}_1 \boldsymbol{\sigma} - \boldsymbol{\Lambda}_2 \boldsymbol{\sigma}$ or $\breve{\boldsymbol{\sigma}} = \mathbf{L}\boldsymbol{\sigma}$. In general, the vectors $\boldsymbol{\sigma}$ and $\breve{\boldsymbol{\sigma}}$ are non-orthogonal. On the other hand, $\breve{\sigma}_{11} = \sigma_{22}$, $\breve{\sigma}_{22} = \sigma_{11}$, $\breve{\sigma}_{12} = -\sigma_{12}$, which means that tensor $\breve{\boldsymbol{\sigma}}$ is the image of tensor $\boldsymbol{\sigma}$ by rotation of the Cartesian frame by a right angle, see [5].

3. The CLM theorem in its first version

This section concerns static problems of in-plane loaded plates subject to tractions, as described in Section 2. Let us stress: no kinematic boundary conditions are imposed and the additional conditions (2.1) on the traction distribution are assumed.

The first version of the CLM theorem says that the perturbation (translation) of the flexibilities by the tensor (2.24) does not change the state of stress within the 2D body considered, provided that the conditions (2.1), (2.2) are fulfilled and the translated elastic moduli tensor remains positive definite. This introduces some bounds on the translation parameter λ. In this form the CLM theorem has been put forward by JASIUK [10], while in the original paper [5] the assumptions (2.1) have been unnecessarily augmented with the requirement that each integral under the sign of the sum in (2.2) vanishes.

3.1. Case of general anisotropy

Consider a 2D body occupying the domain Ω and subjected to the tractions satisfying (2.1), (2.2). The flexibilities of the body are translated by tensor $\mathbf{L}_\lambda$, see (2.24), i.e.

$$\bar{c}_{ijkl} = c_{ijkl} + (\mathbf{L}_\lambda)_{ijkl}. \tag{3.1}$$

The density of energy equals now

$$\bar{W}(\boldsymbol{\tau}(x)) = \frac{1}{2}\boldsymbol{\tau}(x) \cdot (\bar{\mathbf{c}}(x)\boldsymbol{\tau}(x)) \tag{3.2}$$

and, due to (2.26)

$$\bar{W}(\boldsymbol{\tau}) = W(\boldsymbol{\tau}) + \frac{1}{2\lambda}\det\boldsymbol{\tau}. \tag{3.3}$$

The determinant $\det\boldsymbol{\tau}$ can be expressed by using the Airy stress function:

$$2\det\boldsymbol{\tau} = F_{,11}F_{,22} - F_{,12}F_{,21} + F_{,22}F_{,11} - F_{,21}F_{,12}. \tag{3.4}$$

By using the identities

$$\begin{aligned}
F_{,11}F_{,22} &= (F_{,11}F_{,2})_{,2} - F_{,112}F_{,2}\,,\\
F_{,12}F_{,21} &= (F_{,12}F_{,2})_{,1} - F_{,121}F_{,2}\,,\\
F_{,22}F_{,11} &= (F_{,22}F_{,1})_{,1} - F_{,221}F_{,1}\,,\\
F_{,21}F_{,12} &= (F_{,21}F_{,1})_{,2} - F_{,122}F_{,1}\,,
\end{aligned} \tag{3.5}$$

we find

$$2\det\boldsymbol{\tau} = (F_{,11}F_{,2})_{,2} - (F_{,12}F_{,2})_{,1} + (F_{,22}F_{,1})_{,1} - (F_{,21}F_{,1})_{,2}\,, \tag{3.6}$$

hence

$$2\int_{\Omega} \det \boldsymbol{\tau}\, dx = \int_{\Gamma} \big[F_{,2}\,(F_{,11}n_2 - F_{,12}n_1) + F_{,1}\,(F_{,22}n_1 - F_{,12}n_2)\big]\, ds \tag{3.7}$$
$$= \int_{\Gamma} \big[F_{,2}\,(\tau_{22}n_2 + \tau_{21}n_1) + F_{,1}\,(\tau_{11}n_1 + \tau_{12}n_2)\big]\, ds$$
$$= \int_{\Gamma} \big[F_{,2}\,T_2 + F_{,1}\,T_1\big]\, ds = \sum_{i=0}^{n} \int_{\Gamma_i} \nabla F \cdot \mathbf{T} ds_i$$

and, finally

$$2\int_{\Omega} \det \boldsymbol{\tau}\, dx = \sum_{i=0}^{n} J_i(\mathbf{T}) \tag{3.8}$$

because, as proved in Section 2, the integrals J_i given by (2.13) do not depend on the function F, hence J_i depend only on the geometry of the domain Ω and on the traction distribution. Let us come back to (3.3). Integration gives

$$\int_{\Omega} \bar{W}(\boldsymbol{\tau})\, dx = \int_{\Omega} W(\boldsymbol{\tau})\, dx + \frac{1}{4\lambda} \sum_{i=0}^{n} J_i(\mathbf{T}). \tag{3.9}$$

Let

$$\bar{\Im}(\boldsymbol{\tau}) = \int_{\Omega} \bar{W}(\boldsymbol{\tau}(x))\, dx \tag{3.10}$$

and let the field $\bar{\boldsymbol{\sigma}}$ be the stress field arising in the body with the translated flexibilities (3.1). Such a stress field solves the problem:

$$\bar{\Im}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\tau} \in \Sigma_T(\Omega)} \int_{\Omega} \bar{W}(\boldsymbol{\tau})\, dx \tag{3.11}$$

or

$$\bar{\Im}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\tau} \in \Sigma_T(\Omega)} \int_{\Omega} W(\boldsymbol{\tau})\, dx + \frac{1}{4\lambda} \sum_{i=0}^{n} J_i(\mathbf{T}). \tag{3.12}$$

Consequently,

$$\bar{\Im}(\bar{\boldsymbol{\sigma}}) = \Im(\boldsymbol{\sigma}) + \frac{1}{4\lambda} \sum_{i=0}^{n} J_i(\mathbf{T}) \tag{3.13}$$

and $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$, since the minimization operation in (3.12) gives a unique result. Thus, we conclude that the stress field which appears in the body of flexibilities $\bar{c}_{ijkl}$ coincides with the stress field arising in the body with flexibilities c_{ijkl}. In particular, the constant λ does not affect the stress field. The mentioned identity of the stress fields is conditioned by the lower and upper bounds to be imposed on the value of the constant λ to assure positive definiteness of the flexibility tensor (3.1).

3.2. Case of isotropy. J.H. Michell's theorem

Consider now the isotropy case to reveal the range of admissible values of the constant λ. Let tensor $\mathbf{C}(x)$ be represented by (2.17) with positive bulk and shear moduli. Then, the tensor of flexibilities $\mathbf{c}(x)$ has the form (2.21) while the translated flexibility tensor is, according to (3.1), (2.24), given by

$$\bar{\mathbf{c}} = \frac{1}{2\bar{k}(x)}\boldsymbol{\Lambda}_1 + \frac{1}{2\bar{\mu}(x)}\boldsymbol{\Lambda}_2, \tag{3.14}$$

where

$$\frac{1}{\bar{k}(x)} = \frac{1}{k(x)} + \frac{1}{\lambda}, \qquad \frac{1}{\bar{\mu}(x)} = \frac{1}{\mu(x)} - \frac{1}{\lambda}. \tag{3.15}$$

The constant λ satisfies the bounds

$$-\frac{1}{k(x)} < \frac{1}{\lambda} < \frac{1}{\mu(x)} \qquad \forall x \in \Omega. \tag{3.16}$$

Thus, the stress fields arising in the 2D bodies of moduli $\bar{k}(x)$, $\bar{\mu}(x)$ given by (3.15), with the conditions (3.16) fulfilled, are not affected by the value of the constant λ.

The theorem by Michell [1] holds: the stress field does not depend on the values of both the moduli of isotropy, if they are constant within the domain Ω. Indeed, the functional $\bar{W}(\boldsymbol{\tau})$ has the form

$$\bar{W}(\boldsymbol{\tau}) = \frac{1}{E}(\operatorname{Tr}\boldsymbol{\tau})^2 + \left(\frac{1}{2\lambda} - \frac{1}{2\mu}\right)\det\boldsymbol{\tau}, \tag{3.17}$$

where Young's modulus E is the harmonic mean of $2k$, 2μ, or

$$\frac{1}{E} = \frac{1}{4k} + \frac{1}{4\mu}, \tag{3.18}$$

see (2.20). If both the moduli k and μ are constant, then the minimizer of (3.11) coincides with the minimizer of the problem

$$\min_{\boldsymbol{\tau}\in\Sigma_T(\Omega)} \int_{\Omega} (\operatorname{Tr}\boldsymbol{\tau})^2\, dx, \tag{3.19}$$

which does not involve any moduli. This is true only if the traction load satisfies the conditions (2.1)–(2.2). Let us remind that if the domain Ω is simply connected, these conditions reduce to the inevitable conditions of the load being self-equilibrated.

4. The CLM theorem in its second version

The second version of the CLM theorem concerns the theory of homogenization of moduli in the 2D elasticity. In the basic cell problem (the basic cell being the rectangular domain $\Omega = Y = (0, l_1) \times (0, l_2)$) the boundary load assumes opposite values on the opposite sides of the cell. In this version the CLM theorem says that the translation of the flexibilities by tensor (2.24) results in the same translation of the effective flexibilities, see [5]. In this problem the assumptions (2.1) for $i \geq 1$ are identically fulfilled, since in the basic cell problem the internal contours of the basic cell are free of loads.

4.1. Case of arbitrary anisotropy of the basic cell

The CLM theorem plays an essential role in the theory of homogenization of Y-periodic media. A complete proof of this theorem is not easy available. It is reported here for the reader's convenience.

In the stress-based theory of homogenization the test stress fields within the basic cell are subject to the following conditions:

$$
\begin{aligned}
&\tau_{ij},_j = 0 \quad \text{in } Y, \qquad (\ ),_i = \partial(\ )/\partial y_i,\\
&\tau_{ij} n_i \text{ assume opposite values at opposite sides of } Y,\\
&\tau_{ij} n_i = 0 \text{ along the contours of openings within } Y.
\end{aligned}
\tag{4.1}
$$

Here $\mathbf{n} = (n_1, n_2)$ is the unit vector outward normal to the contour of Y.

Such fields form the set $\Sigma_{per}(Y)$. In the theory of homogenization the following equality is proved:

$$
\langle \boldsymbol{\tau} \cdot (\mathbf{L}\boldsymbol{\tau}) \rangle = \langle \boldsymbol{\tau} \rangle \cdot (\mathbf{L}\langle \boldsymbol{\tau} \rangle) \qquad \forall \boldsymbol{\tau} \in \Sigma_{per}(Y),
\tag{4.2}
$$

where tensor $\mathbf{L}$ is given by (2.24) and $\langle \cdot \rangle$ is the averaging operation over the cell Y, or

$$
\langle f \rangle = \frac{1}{|Y|} \int_Y f \, dy, \qquad |Y| = l_1 l_2, \qquad dy = dy_1 \, dy_2.
\tag{4.3}
$$

The equality (4.2) looks like an algebraic formula, but its simplicity is misleading; in the process of proving (4.2) the differential equilibrium equations $(4.1)_1$ play a crucial role; for the complete proof the reader is referred to Sec. 21.4.5 in Lewiński and Telega [11].

The effective flexibilities c^h_{ijkl} of a Y-periodic composite of local flexibilities $c_{ijkl}(y)$ are given by the formula of Suquet [12]:

$$\hat{\boldsymbol{\sigma}} \cdot (\mathbf{c}^h \hat{\boldsymbol{\sigma}}) = \min\{\langle \boldsymbol{\tau} \cdot (\mathbf{c}\boldsymbol{\tau})\rangle \mid \boldsymbol{\tau} \in \Sigma_{per}(Y),\ \langle \boldsymbol{\tau} \rangle = \hat{\boldsymbol{\sigma}}\}, \tag{4.4}$$

where $\hat{\boldsymbol{\sigma}}$ is an arbitrary tensor from E^2_s, cf. [11, Sec. 3.6]. In a similar manner one can compute the effective flexibilities $\bar{c}^h_{ijkl}$ of the Y-periodic composite of flexibilities given by (3.1):

$$\hat{\boldsymbol{\sigma}} \cdot (\bar{\mathbf{c}}^h \hat{\boldsymbol{\sigma}}) = \min\{\langle \boldsymbol{\tau} \cdot (\bar{\mathbf{c}}\boldsymbol{\tau})\rangle \mid \boldsymbol{\tau} \in \Sigma_{per}(Y),\ \langle \boldsymbol{\tau} \rangle = \hat{\boldsymbol{\sigma}}\}. \tag{4.5}$$

Substitution of (3.1) and taking into account (4.2) results in the formula:

$$\hat{\boldsymbol{\sigma}} \cdot (\bar{\mathbf{c}}^h \hat{\boldsymbol{\sigma}}) = \hat{\boldsymbol{\sigma}} \cdot (\mathbf{c}^h \hat{\boldsymbol{\sigma}}) + \hat{\boldsymbol{\sigma}} \cdot (\mathbf{L}_\lambda \hat{\boldsymbol{\sigma}}) \tag{4.6}$$

which, due to arbitrariness of $\hat{\boldsymbol{\sigma}}$, implies the translation rule

$$\bar{\mathbf{c}}^h = \mathbf{c}^h + \mathbf{L}_\lambda, \tag{4.7}$$

similar to the translation rule (3.1). Moreover, it occurs that the minimizer of the problem (4.4) coincides with the minimizer of the problem (4.5). The stress fields within the basic cell arising in both the composites of different flexibilities are the same.

We have assumed for simplicity that the underlying Y-periodic microstructure is uniform, i.e. x-independent. However, the homogenization result (4.4) extends to the non-uniform case; then the homogenized moduli vary within the domain Ω.

4.2. The case of composites of isotropic effective properties

Despite local anisotropy of properties of the basic cell the effective properties of the composite may be isotropic. For example, rotational symmetry with respect to three-fold rotation by 120° implies effective isotropy, see Lewiński [13], Łukasiak [14] and Czarnecki *et al.* [15]. Then the tensor of effective flexibilities is represented by

$$\mathbf{c}^h = \frac{1}{2k^h}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu^h}\boldsymbol{\Lambda}_2 \tag{4.8}$$

and according to (4.7), the tensor of effective flexibilities of the composite with translated values is given by

$$\bar{\mathbf{c}}^h = \frac{1}{2\bar{k}^h}\boldsymbol{\Lambda}_1 + \frac{1}{2\bar{\mu}^h}\boldsymbol{\Lambda}_2, \tag{4.9}$$

where

$$\frac{1}{2\bar{k}^h} = \frac{1}{2k^h} + \frac{1}{2\lambda}, \qquad \frac{1}{2\bar{\mu}^h} = \frac{1}{2\mu^h} - \frac{1}{2\lambda}. \tag{4.10}$$

Thus,

$$\frac{1}{\bar{k}^h} + \frac{1}{\bar{\mu}^h} = \frac{1}{k^h} + \frac{1}{\mu^h}. \tag{4.11}$$

Within the 2D setting the Young moduli are defined by

$$\frac{4}{E^h} = \frac{1}{k^h} + \frac{1}{\mu^h}, \qquad \frac{4}{\bar{E}^h} = \frac{1}{\bar{k}^h} + \frac{1}{\bar{\mu}^h}, \tag{4.12}$$

hence $E^h = \bar{E}^h$. Thus, the translation of flexibilities (3.1) does not affect the value of Young's modulus. It has an influence on the value of Poisson's ratio.

4.3. Case of a porous isotropic composite

Let us remind the known fact of the theory of homogenization of porous composites, see [10]. Assume that we have two isotropic porous media whose 2D Young moduli and Poisson ratios are: a) E, ν, b) E, $\bar{\nu}$ and $\bar{\nu} \neq \nu$. Assume that their effective properties are isotropic and are expressed by the moduli: a) E^h, ν^h, b) $\bar{E}^h$, $\bar{\nu}^h$. It occurs that the effective Young moduli coincide: $\bar{E}^h = E^h$. To prove this equality let us assume that the constant of translation (2.24) equals

$$\lambda = \frac{E}{2(\nu - \bar{\nu})}. \tag{4.13}$$

The bulk and shear moduli of both the materials are:

$$k = \frac{E}{2(1-\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \tag{4.14}$$

$$\bar{k} = \frac{E}{2(1-\bar{\nu})}, \qquad \bar{\mu} = \frac{E}{2(1+\bar{\nu})}. \tag{4.15}$$

Hence

$$\frac{1}{\bar{k}} - \frac{1}{k} = \frac{1}{\lambda}, \qquad \frac{1}{\bar{\mu}} - \frac{1}{\mu} = -\frac{1}{\lambda}, \tag{4.16}$$

where λ is given by (4.13). Thus, the tensors $\mathbf{c}$ and $\bar{\mathbf{c}}$ are linked by Eq. (3.1), where $\mathbf{L}_\lambda$ is defined by (2.24), (4.13). According to (4.7), and, consequently, according to (4.10)–(4.12) we arrive at $\bar{E}^h = E^h$, see [10, Sec. 5.1]. The above result paves the way towards a less obvious property of porous composites of isotropic effective layout of the moduli: the effective Young modulus of such composites does not dependent on the value of Poisson's ratio; for the proof see [5, Sec. 4] and [10, Sec. 5].

5. Extension of the CLM theorem to the thin plate bending problem

5.1. The static problem of a thin plate in bending, whose contour is fully clamped and subject to nonhomogeneous kinematic boundary conditions

Consider the problem of statics of an anisotropic thin elastic plate subject to a transverse load; the plate middle plane occupies a plane domain Ω whose contour Γ is composed of the external contour Γ_0 and the internal contours $\Gamma_1, \ldots, \Gamma_n$ of openings. The Kirchhoff theory of thin plates is applied. All the contours are clamped; the kinematic boundary conditions are assumed as inhomogeneous, namely:

$$w|_{\partial\Omega} = w_0(s), \qquad \frac{\partial w}{\partial \mathbf{n}}|_{\partial\Omega} = \beta_0(s) \quad \text{along } \Gamma, \tag{5.1}$$

where $\mathbf{n} = (n_1, n_2)$is the unit vector outward normal to the contour Γ; the unit vector tangent to the contour is denoted by $\mathbf{t} = (t_1, t_2)$, $t_1 = -n_2$, $t_2 = n_1$. Thus, we have

$$\frac{\partial f}{\partial \mathbf{t}} = f_{,1}\, t_1 + f_{,2}\, t_2 = -f_{,1}\, n_2 + f_{,2}\, n_1. \tag{5.2}$$

The functions $w_0(s)$, $\beta_0(s)$ determine the first derivatives of w along the contour Γ:

$$\begin{aligned} w_{,1} &= w_{,\mathbf{n}} n_1 + w_{,\mathbf{t}} t_1 = \beta_0(s) n_1 + \frac{dw_0}{ds}(-n_2), \\ w_{,2} &= w_{,\mathbf{n}} n_2 + w_{,\mathbf{t}} t_2 = \beta_0(s) n_2 + \frac{dw_0}{ds}(n_1). \end{aligned} \tag{5.3}$$

Thus, ∇w is an *a priori* known function along the whole contour of the plate. Let us compute

$$\begin{aligned} \frac{\partial(w_{,1})}{\partial s} &= w_{,11} t_1 + w_{,12} t_2 = -(w_{,11} n_2 - w_{,12} n_1) \overset{\text{df}}{=} -\varphi_2(s), \\ \frac{\partial(w_{,2})}{\partial s} &= w_{,21} t_1 + w_{,22} t_2 = -w_{,21} n_2 + w_{,22} n_1 \overset{\text{df}}{=} \varphi_1(s). \end{aligned} \tag{5.4}$$

Thus, the functions $\varphi_1(s)$, $\varphi_2(s)$ are determined by the functions $w_0(s)$, $\beta_0(s)$ given along the contour Γ.

The kinematically admissible deflections should be appropriately regular within the domain Ω and should satisfy the boundary conditions (5.1). The set of such deflections has been denoted by $V(\Omega)$, see Section 1.

The plate is subject to the transverse load of intensity $q(x)$. The bending stiffnesses D_{ijkl} form the tensor field $\mathbf{D}$ in the domain Ω, locally of the class

E_s^4 and positive definite. The measures of bending are defined by $\kappa_{11} = -w_{,11}$, $\kappa_{22} = -w_{,22}$, $\kappa_{12} = \kappa_{21} = -w_{,12}$. The bending moments are linked with the transverse load by the equilibrium equation

$$M_{11,11} + M_{22,22} + 2M_{12,12} + q = 0, \tag{5.5}$$

while the bending measures satisfy the compatibility equations:

$$\kappa_{22},_1 - \kappa_{12},_2 = 0, \qquad -\kappa_{12},_1 + \kappa_{11},_2 = 0. \tag{5.6}$$

The deflection $w = w(x)$ is the minimizer of the problem (1.2), where now

$$\Pi(v) = \frac{1}{2}\int_{\Omega} \boldsymbol{\kappa}(v)\cdot(\mathbf{D}\boldsymbol{\kappa}(v))\,dx \tag{5.7}$$

and here the notation: $\boldsymbol{\kappa}(v) = (\kappa_{ij}(v) = -v_{,ij})$ is used. This problem is well posed, (see [8, Ch. IV]).

In case of isotropy the bending moments $M_{ij}$are linked with the measures of bending by

$$M_{11} = D(\kappa_{11} + \nu\kappa_{22}), \qquad M_{22} = D(\nu\kappa_{11} + \kappa_{22}), \qquad M_{12} = D(1-\nu)\kappa_{12}, \tag{5.8}$$

D is treated as independent of ν. The energy stored in the thin isotropic plate is expressed by the functional whose argument is the plate deflection:

$$\Pi(w) = \frac{1}{2}\int_{\Omega} D\Big[(\kappa_{11}(w))^2 + 2\nu\kappa_{11}(w)\kappa_{22}(w) + (\kappa_{22}(w))^2 + 2(1-\nu)(\kappa_{12}(w))^2\Big]\,dx. \tag{5.9}$$

The formula above can be re-written in the form

$$\Pi(w) = \frac{1}{2}\int_{\Omega} D\Big[(\kappa_{11}(w) + \kappa_{22}(w))^2 - \underline{2(1-\nu)(\kappa_{11}(w)\kappa_{22}(w) - (\kappa_{12}(w))^2)}\Big]\,dx \tag{5.10}$$

equivalent to (1.1).

The static-geometric analogy

The *static-geometric analogy* concerns the system of equations of both the static problems of: in-plane loaded and transversely loaded plates. Let us note that by replacing $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ with $(\kappa_{22}, \kappa_{11}, -\kappa_{12})$ and by replacing $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$

with $(M_{22}, M_{11}, -M_{12})$ one rearranges: Eqs. (2.3) (written in terms of $\sigma_{11}, \sigma_{22}, \sigma_{12}$ components), and (2.9) to the form of Eqs. (5.6), (5.5), if $q = 0$. Thus, the equilibrium equations of the in-plane loaded plate assume the form of the compatibility equations for bending measures, while the compatibility equation of the in-plane loaded plate assumes the form of the equilibrium equation of the bending plate, if $q = 0$. Moreover, the expressions for the elastic energies: (2.23) and (5.10) are similar. To make them identical one should replace $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ with $(\kappa_{22}, \kappa_{11}, -\kappa_{12})$ and then replace (ν, E) with $(-\nu, 1/D)$, respectively. Moreover, by expressing the stresses by using the Airy function F, see (2.10), we rearrange the formula (2.23) to the form (1.1). This analogy clears up why the CLM theorem applies to controlling: the flexibilities of in-plane plates and – the bending stiffnesses of transversely loaded plates.

It is remarkable and surprising that the static-geometric analogy takes place in some selected theories of thin elastic shells, as discovered by A. L. Goldenveizer in 1940's. The class of such theories has been identified in BUDIANSKY and SANDERS [16].

5.2. The CLM theorem in its first version

Consider now the plate of the same shape and loaded the same way but of different stiffnesses, translated according to the rule:

$$\bar{\mathbf{D}}(x) = \mathbf{D}(x) + \mathbf{L}_\lambda, \tag{5.11}$$

where $\mathbf{L}_\lambda$ is given by (2.24), yet note that now the modulus $1/\lambda$ has the dimension of a bending stiffness. By analogy with (3.3) we find

$$\frac{1}{2}\boldsymbol{\kappa} \cdot (\bar{\mathbf{D}}\boldsymbol{\kappa}) = \frac{1}{2}\boldsymbol{\kappa} \cdot (\mathbf{D}\boldsymbol{\kappa}) + \frac{1}{2\lambda} \det \boldsymbol{\kappa}, \tag{5.12}$$

where now, see (3.4),

$$2 \det \boldsymbol{\kappa} = w_{,11}w_{,22} - w_{,12}w_{,21} + w_{,22}w_{,11} - w_{,21}w_{,12}. \tag{5.13}$$

By virtue of the analogy between (3.4) and (5.13) we find

$$2\int_\Omega \det \boldsymbol{\kappa}\, dx = \int_\Gamma \Big[w_{,2}\,(w_{,11}n_2 - w_{,12}n_1) + w_{,1}\,(w_{,22}n_1 - w_{,12}n_2)\Big]\, ds \tag{5.14}$$

and, using (5.4) we reduce the integral as follows

$$\begin{aligned} 2\int_\Omega \det \boldsymbol{\kappa}\, dx &= \int_\Gamma \left[w_{,2}\left(-\frac{\partial}{\partial s}(w_{,1})\right) + w_{,1}\left(\frac{\partial}{\partial s}(w_{,2})\right)\right] ds \\ &= \int_\Gamma \nabla w \cdot \boldsymbol{\varphi}\, ds, \end{aligned} \tag{5.15}$$

where both: ∇w and $\boldsymbol{\varphi}$ are determined by the functions $w_0(s)$, $\beta_0(s)$ given along the contour. This confirms the conjecture noted in Section 1: $\frac{1}{2}\nabla w \cdot \boldsymbol{\varphi}$ can be viewed as approximation of the geodesic curvature of the contour of the deformation surface.

The deflection of the plate of stiffnesses $\bar{D}_{ijkl}$ is the minimizer of the problem

$$\min_{v \in V(\Omega)} \left\{ \bar{\Pi}(v) - \int_\Omega qv\,dx \right\} \tag{5.16}$$

with

$$\bar{\Pi}(v) = \Pi(v) + \underline{\frac{1}{4\lambda} \int_\Gamma \nabla v \cdot \boldsymbol{\varphi}\,ds}, \tag{5.17}$$

where the underlined term is determined only by the functions $w_0(s)$, $\beta_0(s)$ given along the contour. We conclude that the minimizers of both the problems (1.2), (5.7) and (5.16) are the same: $\bar{w} = w$. Consequently, the bending deformation measures coincide. The bending moments are different:

$$\begin{aligned} \mathbf{M} &= \mathbf{D}\boldsymbol{\kappa}(w), \\ \bar{\mathbf{M}} &= \mathbf{M} + \frac{1}{2\lambda}(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)\boldsymbol{\kappa}(w). \end{aligned} \tag{5.18}$$

5.3. Homogenization of stiffnesses of thin elastic Y-periodic plates in bending

5.3.1. Translation of the bending flexibilities. Consider a thin elastic Y-periodic plate, i.e. of Y-periodically varying flexibilities $d_{ijkl}(y)$. The test bending moment fields $m_{ij}(y)$ within the basic cell satisfy the conditions, see [11, Sec. 3.6]:

$$\begin{aligned} & m_{ij,ij} = 0 \text{ in } Y, \quad \text{here } (.)_{,i} = \partial(.)/\partial y_i, \\ & m_n = m_{ij} n_i n_j \quad \text{assume the same values at the opposite sides of } Y, \\ & q_n = n_i m_{ij},_j + \frac{\partial}{\partial s}(m_{ij} n_i t_j) \\ & \qquad\qquad\qquad \text{assume opposite values at the opposite sides of } Y. \end{aligned} \tag{5.19}$$

Here the vector $\mathbf{t}$ is a unit vector tangent to the contour ∂Y. Such fields $m_{ij}(y)$ form the set $S_{per}(Y)$. The effective flexibilities d^h_{ijkl} of the plate are given by the variational formula

$$\mathbf{M} \cdot (\mathbf{d}^h \mathbf{M}) = \min\big\{ \langle \mathbf{m} \cdot (\mathbf{dm}) \rangle \mid \mathbf{m} \in S_{per}(Y),\ \langle \mathbf{m} \rangle = \mathbf{M} \big\}, \tag{5.20}$$

see [11, Eq. (3.6.7) in Sec. 3.6].

If $\mathbf{m} = (m_{ij}) \in S_{per}(Y)$ then

$$\langle \mathbf{m} \cdot (\mathbf{L}\mathbf{m}) \rangle \leq \langle \mathbf{m} \rangle \cdot (\mathbf{L}\langle \mathbf{m} \rangle) \tag{5.21}$$

see [11, Eq. (21.4.13)].

Along with the problem (5.20) concerning plates of flexibilities $d_{ijkl}(y)$ let us consider the plate of the same shape, equally loaded, of flexibilities translated by the rule (3.1), or

$$\bar{\mathbf{d}}^h = \mathbf{d}^h + \mathbf{L}_\lambda. \tag{5.22}$$

Thus, by (2.26) we compute

$$\begin{aligned} \mathbf{m} \cdot (\bar{\mathbf{d}}\mathbf{m}) &= \mathbf{m} \cdot (\mathbf{d}\mathbf{m}) + \mathbf{m} \cdot (\mathbf{L}_\lambda \mathbf{m}) = \mathbf{m} \cdot (\mathbf{d}\mathbf{m}) + \frac{1}{2\lambda}\mathbf{m} \cdot (\mathbf{L}\mathbf{m}) \\ &= \mathbf{m} \cdot (\mathbf{d}\mathbf{m}) + \frac{1}{\lambda} \det \mathbf{m}. \end{aligned} \tag{5.23}$$

If $\lambda > 0$, by virtue of (5.21), we have

$$\langle \mathbf{m} \cdot (\bar{\mathbf{d}}\mathbf{m}) \rangle \leq \langle \mathbf{m} \cdot (\mathbf{d}\mathbf{m}) \rangle + \frac{1}{2\lambda} \langle \mathbf{m} \rangle \cdot (\mathbf{L}\langle \mathbf{m} \rangle). \tag{5.24}$$

Thus

$$\mathbf{M} \cdot (\bar{\mathbf{d}}^h \mathbf{M}) \leq \mathbf{M} \cdot (\mathbf{d}^h \mathbf{M}) + \mathbf{M} \cdot (\mathbf{L}_\lambda \mathbf{M}) \quad \forall \mathbf{M} \in E_s^2 \tag{5.25}$$

or tensor $\bar{\mathbf{d}}^h - \mathbf{d}^h - \mathbf{L}_\lambda$ is negative definite; there is no argument to claim that $\bar{\mathbf{d}}^h = \mathbf{d}^h + \mathbf{L}_\lambda$.

5.3.2. Translation of the bending stiffness tensor. Let $\mathbf{D} = \mathbf{d}^{-1}$ be a Y-periodic tensor of stiffnesses of a thin plate. We consider the plate of the same shape, equally loaded, of Y-periodic stiffnesses given by the translation rule:

$$\bar{\mathbf{D}} = \mathbf{D} + \mathbf{L}_\lambda. \tag{5.26}$$

The effective bending stiffnesses of the Y-periodic plate of stiffnesses D_{ijkl} are determined by the variational formula, see [11, Eq. (3.4.10)]:

$$\mathbf{k} \cdot (\mathbf{D}^h \mathbf{k}) = \min\left\{ \langle \boldsymbol{\kappa} \cdot (\mathbf{D}\boldsymbol{\kappa}) \rangle \mid \boldsymbol{\kappa} \in K_{per}(Y),\ \langle \boldsymbol{\kappa} \rangle = \mathbf{k} \right\} \quad \forall \mathbf{k} \in E_s^2, \tag{5.27}$$

where the set $K_{per}(Y)$ is composed of kinematically admissible bending deformation measures κ_{ij} satisfying the compatibility equation (5.6), where now $\partial(.)/\partial y_i = (.)_{,i}$; these bending measures are associated with Y-periodic deflections of the basic cell. The equality

$$\langle \det \boldsymbol{\kappa} \rangle = \det \langle \boldsymbol{\kappa} \rangle \quad \forall \boldsymbol{\kappa} \in K_{per}(Y) \tag{5.28}$$

(see [11, Eq. (21.4.4)) implies the rule:

$$\langle \boldsymbol{\kappa} \cdot (\mathbf{L}\boldsymbol{\kappa})\rangle = \langle \boldsymbol{\kappa}\rangle \cdot (\mathbf{L}\langle \boldsymbol{\kappa}\rangle) \forall \boldsymbol{\kappa} \in K_{per}(Y). \tag{5.29}$$

It is worth noting that this formula has the same form as Eq. (4.2) concerning the theory of in-plane loaded plates.

The effective stiffnesses $\bar{D}^h_{ijkl}$of the plate of Y-periodic stiffnesses $\bar{D}_{ijkl}$ given by (5.26) are determined by the formula (5.27) or

$$\mathbf{k} \cdot (\bar{\mathbf{D}}^h \mathbf{k}) = \min\left\{ \langle \boldsymbol{\kappa} \cdot (\bar{\mathbf{D}}\boldsymbol{\kappa})\rangle \mid \boldsymbol{\kappa} \in K_{per}(Y),\, \langle \boldsymbol{\kappa}\rangle = \mathbf{k}\right\} \quad \forall \mathbf{k} \in E^2_s. \tag{5.30}$$

Substitution of (5.26), taking into account (5.29), gives

$$\mathbf{k} \cdot (\bar{\mathbf{D}}^h \mathbf{k}) = \mathbf{k} \cdot (\mathbf{D}^h \mathbf{k}) + \mathbf{k} \cdot (\mathbf{L}_\lambda \mathbf{k}) \quad \forall \mathbf{k} \in E^2_s, \tag{5.31}$$

hence

$$\bar{\mathbf{D}}^h = \mathbf{D}^h + \mathbf{L}_\lambda. \tag{5.32}$$

We conclude that the above translation of the effective bending stiffnesses is compatible with the point-wise translation (5.26) of the local bending stiffnesses.

6. Final remarks

The Cherkaev–Lurie–Milton theorem applies both to the theory of in-plane loaded plates and transversely loaded plates, which can be justified by the static-geometric analogy. From the mathematical viewpoint the Cherkaev–Lurie–Milton theorem may be treated as a by-product of the theory of null-lagrangians, see Cherkaev [17]. On the other hand, its role in the theory of composites is so strong and treated as obvious that the proof of this theorem is usually omitted, overshadowed by the closely related problems of bounding the moduli, bounding the energy and other subtle optimum design questions. Indeed, the theorem has found its vital applications in optimum design of: two-phase and three-phase composite bodies, see Jasiuk [10], Cherkaev [17], Milton [18], Cherkaev and Dzierżanowski [19], and in the problems of optimization of composite plates, see Lurie and Cherkaev [4], cf. [11], and composite shells, see Dzierżanowski [20].

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